Fractional Distance: The Topology of the Real Number Line with Applications to the Riemann Hypothesis

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Abstract

Recent analysis has uncovered a broad swath of rarely considered real numbers called real numbers in the neighborhood of infinity. Here we extend the catalog of the rudimentary analytical properties of all real numbers by defining a set of fractional distance functions on the real number line and studying their behavior. The main results are (1) to prove with modest axioms that some real numbers are greater than any natural number, (2) to develop a technique for taking a limit at infinity via the ordinary Cauchy definition reliant on the classical epsilon-delta formalism, and (3) to demonstrate an infinite number of non-trivial zeros of the Riemann zeta function in the neighborhood of infinity. We define numbers in the neighborhood of infinity as Cartesian products of Cauchy equivalence classes of rationals. We axiomatize the arithmetic of such numbers, prove the operations are well-defined, and then make comparisons to the similar axioms of a complete ordered field. After developing the many underling foundations, we present a basis for a topology.

Contents

§1 Introduction

The original Euclidean definition of a real number [1] has given way over time to newer constructive definitions such as the Cauchy equivalence class suggested by Cantor [2], the Dedekind cut [3], and also axiomatic definitions, the most popular of which are the axioms of a complete ordered field based in Hilbert's axioms of geometry [4]. The main purpose of the present analysis is to compare and contrast geometric and algebraic constructions of the real numbers, and then to give a hybrid constructive-axiomatic definition which increases the mutual complements among the two notions of geometry and algebra.

Throughout most of the history of mathematics, it was sufficient to give the Euclidean geometric conception of numbers as cuts in an infinite line, or "magnitudes" as Euclid is usually translated [1]. The Euclid definition of $\mathbb{R}$ has its foundation in physical measurement. In modernity, the preoccupation of mathematics with algebra more so than quantity has stimulated the development of alternatives which are said to be "more rigorous" than Euclid. The main development of this present fractional distance analysis is to present an alternative set of algebraic constructions and axioms which more thoroughly preserve the geometric notion that a number is a cut in an infinite line. We will show that Cantor's definition of $\mathbb{R}$ as the set of all Cauchy equivalence classes of rationals leaves something to be desired with respect to the underlying conception of $\mathbb{R}$ as an open-ended infinite line $(-\infty, \infty)$. The equivalence class construction of $\mathbb{R}$, which is based on an assumed set of rational numbers $\mathbb{Q}$, precludes the existence of a neighborhood of infinity distinct from any neighborhood of the origin, as does the similar Dedekind cut.

For a finite interval $x' \in [0, \frac{\pi}{2})$, we may use $x = \tan(x')$ to construct the interval $x \in [0, \infty)$ wherein everything is usually considered to be a real number. We will develop the notion of fractional distance to prove that if there exists a number at the Euclidean midpoint $x' = \frac{\pi}{4}$ of $[0, \frac{\pi}{2})$, then the bijectivity of the tangent function $f(x) = \tan(x)$ on $[0, \frac{\pi}{2})$ should require a real number at the Euclidean midpoint of $[0, \infty)$. A proof (Theorem 3.2.2) that there must exist such a number is the linchpin of everything in this analysis. Indeed, since Euler himself used this number [5–7], calling it $\frac{i}{2}$ in his own work, the fractional distance approach to $\mathbb{R}$ presented here should be considered ***a return to the old rather than a proposition for something new***. Such a number as $\frac{i}{2}$ will be said to be a number in the neighborhood of infinity because it will have non-zero "fractional distance" with respect to infinity. In that regard, we will say that every number having zero fractional distance with respect to infinity is a number in the neighborhood of the origin. We will show that the existence of the neighborhood of infinity is required to preserve Euclid's conception of a number as a cut in an infinite line. We will argue that any construction which preserves the concept of real numbers as cuts in an infinite line is necessarily better than one which overwrites that concept.

Treatment of the neighborhood of infinity as a distinct numerical mode with separate behavior from the neighborhood of the origin is the direct motivator for everything new reported here. We will posit one very modest change to Cantor's Cauchy equivalence class construction such that it will more fully preserve the favorable notion that $\mathbb{R} = (-\infty, \infty)$. This notion is perfectly equivalent to granting that $\mathbb{R}$ has the usual topology. The modified equivalence class construction will give formal constructions for real numbers in the neighborhood of infinity rather than preclude their existence. With our new constructions and axioms given, we will present an analysis of $\mathbb{R}$ yielding unexpected properties which are non-trivial and exciting, and then we will give the formal topology.

In previous work [8,9], we have demonstrated the existence of a broad class of real numbers: those in the neighborhood of infinity. For the present analysis, we will again demonstrate the existence of real numbers in the neighborhood of infinity. Then we will construct such numbers more or less directly from $\mathbb{Q}$, and then we will axiomatize the arithmetic of such numbers and study the consequences which follow.

The structure is as follows.

- Section Two: We give a simple Euclidean definition for real numbers. These geometric considerations set the stage for the algebraic considerations which follow.
- Section Three: We define and analyze a set of functions called fractional distance functions. These functions constitute the kernel of the analytical direction of the present work.
- Section Four: We give the properties of real numbers in the neighborhood of infinity. The ***formal algebraic construction*** of such numbers by Cauchy sequences is given therein.
- Section Five: We axiomatize a set of arithmetic operations for $\mathbb{R}$ and make a comparison with the similar field axioms. We find they are mostly the same but slightly different.
- Section Six: We prove some results with the present arithmetic axioms. Interestingly, we develop a technique by which it is possible to take a limit at infinity with the ordinary Cauchy prescription for limits: something that has been considered heretofore impossible.
- Section Seven: This section is dedicated most specifically to the topological and generally set theoretical properties of the real number line. The main thrust is to define a Cantor-like set on $\mathbb{R}$ and then to examine its consequences for the least upper bound property of connected sets.
- Section Eight: We apply the notions and consequences of fractional distance to the Riemann hypothesis. We show that the Riemann ζ function *does* have non-trivial zeros off the critical line.

§2 Mathematical Preliminary

§2.1 Real Numbers

In this section, the reader is invited to recall the distinction between the real numbers $\mathbb{R}$ and the real ordered number field $\mathcal{R} = \{\mathbb{R}, +, \times, \leq\}$. Real numbers exist independently of their operations. Here, we define real numbers as cuts in the real number line pending a more formal, complementary definition by Cauchy sequences in Section 4, and by Dedekind cuts in Section 7. By defining a line, giving it a label "real," defining cuts in a line, and then defining real numbers as cuts in the real number line, we make a rigorous definition of real numbers sufficient for applications at any level of rigor. Specifically, the definition given in this section underpins the Cauchy and Dedekind definitions given later.

Generally, the definition of real numbers given in the present section is totally equivalent to the Euclidean magnitude defined in Euclid's *Elements*. Fitzpatrick, the translator of Euclid's original Greek in Reference [1], points out that Euclid's analysis was deliberately restricted to that which may be measured with a physical compass and straight edge: what are called the constructible numbers. Euclid surely was well aware, however, that the real number line is of immeasurable, non-constructible length, and that non-constructible numbers exist. The main motivator for the new formalism presented here is that we would like to consider both measurable and immeasurable magnitudes, or constructible and non-constructible numbers, exceeding those which can be defined in the canonical Cauchy and Dedekind approaches [2, 3].

Definition 2.1.1 A line is a 1D Hausdorff space extending infinitely far in both directions. The interval representation of a line is $(-\infty, \infty)$. In other words, the connected interval $(-\infty, \infty)$ is an infinite line.

Definition 2.1.2 A number line is a line equipped with a chart x and the Euclidean metric

$$d(x, y) = \big|y - x\big| \quad .$$

Definition 2.1.3 The real number line is a number line given the label "real."

Definition 2.1.4 If x is a cut in a line, then

$$(-\infty, \infty) = (-\infty, x] \cup (x, \infty) \quad .$$

Definition 2.1.5 A real number $x \in \mathbb{R}$ is a cut in the real number line.

Axiom 2.1.6 Real numbers are such that

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x \neq y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad \big|y - x\big| > \frac{1}{n} \quad .$$

Neither infinitesimals nor numbers having infinitesimal parts are real numbers.

Axiom 2.1.7 Real numbers are represented in algebraic interval notation as

$$\mathbb{R} = (-\infty, \infty) \quad .$$

In other words, $x \in \mathbb{R}$ if x is both less than infinity and greater than minus infinity. The connectedness of $\mathbb{R}$ is explicit in the interval notation.

Remark 2.1.8 In Section 4.2, we will supplement Axiom 2.1.7 by giving a definition in terms of Cauchy equivalence classes. Axiom 2.1.7 is often considered as lacking sufficient rigor but the Cauchy definition will remedy any so-called insufficiencies of the broad generality of Axiom 2.1.7.

Definition 2.1.9 $\mathbb{R}_0$ is a subset of all real numbers

$$\mathbb{R}_0 = \big\{ x \in \mathbb{R} \;\big|\; (\exists n \in \mathbb{N})[-n < x < n] \big\} \quad .$$

Here we define $\mathbb{R}_0$ as the set of all $x \in \mathbb{R}$ such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$. We call this the set of real numbers less than some natural number (where absolute value is implied.) These numbers are said to lie within the neighborhood of the origin.

Definition 2.1.10 $\mathbb{R}_\infty$ is a subset of all real numbers with the property

$$\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0 \quad .$$

§2.2 Affinely Extended Real Numbers

To prove in Section 3.2 that $\mathbb{R}_\infty$ is not the empty set, namely that there are real numbers larger than every natural number, we will make reference to "line segments" beyond the simpler construction called "a line." Most generally, a line with two different endpoints A and B is a called a line segment AB. We will use notation such that $AB \equiv [a, b]$ where $[a, b]$ is an interval of numbers. Nowhere will we require that the endpoints must be real numbers so the interval $[a, b] = [-\infty, \infty]$ will conform to the definition of a line segment. The real line $\mathbb{R}$ together with two endpoints $\{\pm\infty\}$ is called the affinely extended real number line $\overline{\mathbb{R}} = [-\infty, \infty]$. The present section lays the foundation for an analysis of general line segments in Section 2.3 by first giving some properties of $\overline{\mathbb{R}}$.

Definition 2.2.1 For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have the properties

$$\lim_{x \to 0^\pm} \frac{1}{x} = \text{diverges} \quad , \qquad \text{and} \qquad \lim_{n \to \infty} \sum_{k=1}^{n} k = \text{diverges} \quad .$$

Definition 2.2.2 Define two affinely extended real numbers $\pm\infty$ such that for $x \in \mathbb{R}$ and $n, k \in \mathbb{N}$, we have the properties

$$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty \quad , \quad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \infty \quad .$$

The limit as x approaches zero shall be referred to as "the limit definition of infinity."

Axiom 2.2.3 The infinite element ∞ is such that

$$\infty - \infty = \text{undefined} \quad , \quad \text{and} \qquad \frac{\infty}{\infty} = \text{undefined} \quad .$$

Definition 2.2.4 The set of all affinely extended real numbers is

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} \quad .$$

This set is defined in interval notation as

$$\overline{\mathbb{R}} = [-\infty, \infty] \quad .$$

Remark 2.2.5 If $x_n > 0$ with $\{x_n\}$ being a monotonic sequence, the ∞ symbol is such that if $x_n \in \mathbb{R}$, and if

$$\lim_{n\to\infty} x_n = \text{diverges} \quad ,$$

then for the same $x_n \in \overline{\mathbb{R}}$ we have

$$\lim_{n\to\infty} x_n = \infty \quad .$$

Definition 2.2.6 An affinely extended real number $x \in \overline{\mathbb{R}}$ is $\pm\infty$ or it is a cut in the affinely extended real number line:

$$[-\infty, \infty] = [-\infty, x] \cup (x, \infty] \quad .$$

Theorem 2.2.7 *If $x \in \overline{\mathbb{R}}$ and $x \neq \pm\infty$, then $x \in \mathbb{R}$.*

Proof. Proof follows from Definition 2.2.4.

§2.3 Line Segments

In this section, we review what is commonly understood regarding Euclidean line segments [1]. We begin to develop the relationship between points in a line segment and cuts in a line. During the analyses which follow in the remainder of this work, we will closely examine the differences between cuts and points as a proxy for the fundamental relationship between algebra and geometry. Section 3.3 is dedicated specifically to these distinctions though they are treated throughout this text. The general principle of the distinction between cuts and points is the following. If x is a cut in a line, then

$$(-\infty,\infty)=(-\infty,x]\cup(x,\infty)\quad .$$

If x is a point in a line, then we have a tentative, preliminary understanding that

$$(-\infty,\infty)=(-\infty,x)\cup\{x\}\cup(x,\infty)\quad .$$

Definition 2.3.1 A line segment AB is a line together with two different endpoints $A\neq B$.

Definition 2.3.2 AB is a real line segment if and only if the endpoints A and B bound some subset of the real line $\mathbb{R}=(-\infty,\infty)$.

Definition 2.3.3 Much of the analysis presented here will depend on relationships between geometric and algebraic expressions. The $\equiv$ symbol will be used to denote symbolic equality between geometric and algebraic expressions.

Definition 2.3.4 A real line segment AB is represented in interval notation as $AB\equiv[a,b]$ where a and b are any two affinely extended real numbers $a,b\in\overline{\mathbb{R}}$ such that $a<b$.

Definition 2.3.5 The Euclidean notation AB is called the geometric representation of a line segment. The interval notation $[a,b]$ is called the algebraic representation of a line segment.

Axiom 2.3.6 Line segments have the property that

$$AB=AC\qquad\Longleftrightarrow\qquad B=C\quad .$$

Axiom 2.3.7 Two line segments AB and CD are equal, meaning $AB=CD$, if and only if

$$\frac{AB}{CD}=\frac{CD}{AB}=1\quad .$$

Definition 2.3.8 AB is a special label given to the unique real line segment $AB\equiv[0,\infty]$. We have

$$AB=\mathbf{AB}\qquad\Longleftrightarrow\qquad AB\equiv[0,\infty]\quad .$$

Definition 2.3.9 X is an interior point of AB if and only if

$$X\neq A\quad,\quad X\neq B\quad,\quad\text{and}\qquad X\in AB\quad .$$

Axiom 2.3.10 If X is an interior point of AB, then

$$AB = AX + XB \quad .$$

Axiom 2.3.11 Every geometric point X along a real line segment AB has one and only one algebraic interval representation $\mathscr{X}$. If $\mathscr{X}$ is the algebraic representation of X, then $X \equiv \mathscr{X}$ and $\mathscr{X}$ is a unique subset of $[a, b] \equiv AB$.

Definition 2.3.12 The formal meaning of the relation $AB \equiv [a, b]$ is that a is the least number in the algebraic representation of A, b is the greatest number in the algebraic representation of B, and that every other number x in the algebraic representation of any point in AB has the property $a < x < b$.

Theorem 2.3.13 *If X is an interior point of a real line segment AB, then X has an algebraic interval representation as one or more real numbers.*

Proof. X is an interior point of AB so, by Axiom 2.3.10, we have

$$AB = AX + XB \quad .$$

Since $AB \equiv [a, b]$ and $(a, b) \subset \mathbb{R}$, it follows that the algebraic representation $\mathscr{X}$ of an interior point X is such that

$$x \in \mathscr{X} \qquad \Longrightarrow \qquad a < x < b \quad .$$

For $(a, b) \subset \mathbb{R}$, this inequality is only satisfied by $x \in \mathbb{R}$. The theorem is proven.

Remark 2.3.14 It will be a main result of the fractional distance analysis to show that the infinite length of a line segment such as $\mathbf{AB} \equiv [0, \infty]$ will allow us to put more than one number into the algebraic representation $\mathscr{X}$ of a geometric point X. If a line segment has finite length $L \in \mathbb{R}_0$, we will show that there is at most one real number in the algebraic representation of one its interior points. However, this constraint will vanish in certain cases of $\text{len}(AB)$.

Definition 2.3.15 The algebraic representation $\mathscr{X}$ of a geometric point X lying along a real line segment AB is

$$\mathscr{X} = [x_1, x_2] \quad , \quad \text{where} \qquad x_1, x_2 \in \overline{\mathbb{R}} \quad .$$

The special (intuitive) case of $x_1 = x_2 = x$ gives

$$\mathscr{X} = [x, x] = \{x\} = x \quad .$$

Here, we have expressed $\mathscr{X}$ with included endpoints x_1 and x_2. Most generally, however, an algebraic representation of a geometric point is a single number or it is some interval of numbers, *i.e.*: all variations of (x_1, x_2), $(x_1, x_2]$, and $[x_1, x_2)$ are allowable algebraic representations of X. We do not require that $x_1 \neq x_2$ in all cases.

Remark 2.3.16 A point in a line segment has a representation as a set of numbers, possibly only one number, and it remains to identify the exact relationship between numbers (cuts) and geometric points. The key feature of Definition 2.3.15 is that it allows, provisionally, a many-to-one relationship between cuts in lines (algebraic) and points in line segments (geometric.) In Section 3.3, we will strictly prove that which has been suggested: the algebraic representation of $X \in AB$ is only constrained to be a unique real number for certain cases of AB with finite length.

Definition 2.3.17 If $X \equiv \mathscr{X} = [x_1, x_2]$ with $x_1 \neq x_2$, and if $x \in [x_1, x_2]$, then x is said to be a *possible* algebraic representation of X. If $x_1 = x_2 = x$, then x is said to be *the* algebraic representation of X. If x is the algebraic representation of X, then $x \equiv X$. If x is a possible representation of X, then $x \in X$, *i.e.*: if x is a possible algebraic representation of X, then

$$x \in \mathscr{X} = [x_1, x_2] \equiv X \quad .$$

This statement may be abbreviated as $x \in X$ while $x \equiv X$ specifies the case of $x_1 = x_2$.

Definition 2.3.18 A point C is called a midpoint of a line segment AB if and only if

$$\frac{AC}{AB} = \frac{CB}{AB} = \frac{1}{2} \quad .$$

Alternatively, C is a midpoint of AB if and only if

$$AC = CB \quad , \quad \text{and} \qquad AC + CB = AB \quad .$$

Definition 2.3.19 Hilbert's discarded axiom [4] states the following: any four points $\{A, B, C, D\}$ of a line can always be labeled so that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.

Remark 2.3.20 Hilbert's discarded axiom is discarded not because it wrong but rather because it is implicit in Hilbert's other axioms [4]. It is discarded by redundancy rather than invalidity.

Theorem 2.3.21 *All line segments have at least one midpoint.*

Proof. Let there be a line segment AB and two circles of equal radii centered on the points A and B. Let the two radii be less than AB but great enough

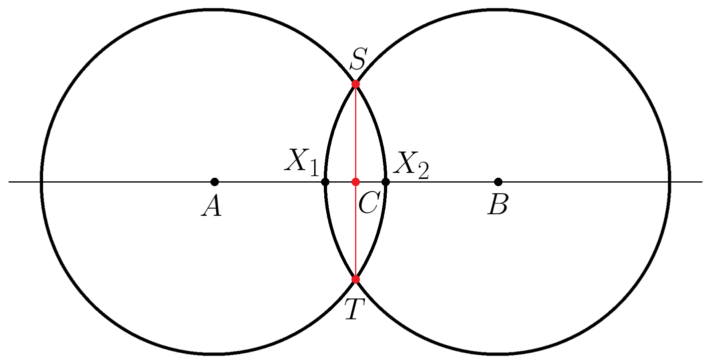

Figure 1: This figure proves that every line segment AB has one and only one midpoint.

such that the circles intersect at exactly two points S and T. The geometric configuration shown in Figure 1 is guaranteed to exist by Hilbert's discarded axiom pertaining to $\{A, X_1, X_2, B\}$. By construction, it follows that

$$AS = AT = BS = BT \quad .$$

Let the line segment ST intersect AB at C. By the Pythagorean theorem, C is a midpoint of AB because

$$AC^2 + CS^2 = AS^2 \quad , \qquad \text{and} \qquad BC^2 + CS^2 = BS^2 \quad ,$$

together yield

$$AC = BC \quad .$$

C separates AB into two line segments so

$$AC + CB = AB \quad .$$

These two conditions, $AC = BC$ and $AC + CB = AB$, jointly conform to Definition 2.3.18 so C is a midpoint of an arbitrary line segment AB. ✎

Example 2.3.22 Theorem 2.3.21 regards an arbitrary line segment AB. Therefore, the theorem holds in the case of an arbitrary line segment AB. One might be afflicted, however, with the assumption that it is not possible to define two such intersecting circles centered on the endpoints of an arbitrary line segment such as $\mathbf{AB} \equiv [0, \infty]$. To demonstrate how the arbitrary case of any line segment AB generalizes to the specific case of $\mathbf{AB}$, let $AB \equiv [0, \frac{\pi}{2}]$ and let $x' \in \mathscr{X}$ be a number in the algebraic representation of $X \in AB$. We say that $[0, \frac{\pi}{2}]$ is the algebraic representation of AB charted in x'. Let x be such that

$$x = \tan(x') \quad ,$$

so that x and x' are two charts related by a conformal transformation. Using

$$\tan(0) = 0 \quad , \qquad \text{and} \qquad \tan\left(\frac{\pi}{2}\right) = \infty \quad ,$$

where the latter follows from Definition 2.2.2, it follows that $[0,\infty]$ is the algebraic representation of AB charted in x. Therefore, $AB = \mathbf{AB}$ with respect to the x chart.

Hilbert's discarded axiom guarantees the existence of two points $X_1 \in AB$ and $X_2 \in AB$ with algebraic representations $\mathscr{X}_1'$ and $\mathscr{X}_2'$ such that, for example,

$$x_1' = \frac{\pi}{6} \in \mathscr{X}_1' \quad , \quad \text{and} \qquad x_2' = \frac{\pi}{3} \in \mathscr{X}_2' \quad .$$

If the radius of the circle centered on A is AX_2 and the radius of the circle centered on B is BX_1, then it is guaranteed that these circles will intersect at two points S and T, as in Figure 1. Since $AB = \mathbf{AB}$ in the x chart, it is required that $X_1 \in \mathbf{AB}$ and $X_2 \in \mathbf{AB}$. Therefore, circles centered on the endpoints of $\mathbf{AB}$ with radii AX_2 and BX_1 will intersect at exactly two points. ***The chart on the line segment cannot affect the line segment's basic geometric properties!*** It is unquestionable that the points X_1 and X_2 exist and are well-defined in the x' chart, and it is not possible to disrupt the geometric configuration by introducing a second chart onto AB. A chart can no more disrupt the geometric configuration than erasing an island from a map might make the physical island disappear from the sea. X_1 and X_2 do not cease to exist simply because we define a conformal chart $x = \tan(x')$. If they ceased to exist, then that would violate Hilbert's discarded axiom. This example demonstrates that Theorem 2.3.21 is valid even for the specific case of the infinite line segment $AB = \mathbf{AB}$.

Theorem 2.3.23 *All line segments have one and only one midpoint.*

Proof. For proof by contradiction, suppose C and D are two different midpoints of a line segment AB. C and D are midpoints of AB so we may derive from Definition 2.3.18

$$AC = CB = \frac{AB}{2} \quad , \quad \text{and} \qquad AD = DB = \frac{AB}{2} \quad .$$

It follows that $AC = AD$. By Axiom 2.3.6, therefore, $C = D$ and we invoke a contradiction having assumed that C and D are different.

§3 Fractional Distance

§3.1 Fractional Distance Functions

If there are two circles with equal radii whose centers are separated by an infinite distance, then what numerical radii less than infinity will allow the circles to intersect at exactly two points? To answer this question, we will introduce fractional distance functions. We will use these functions to demonstrate the existence of real numbers in the neighborhood of infinity.

Definition 3.1.1 For any point X on a real line segment AB, the geometric fractional distance function $\mathcal{D}_{AB}$ is a continuous bijective map

$$\mathcal{D}_{AB}(AX) : AB \to [0,1] \quad ,$$

which takes $AX \subseteq AB$ and returns real numbers. This function returns AX as a fraction of AB. Emphasizing the geometric construction, the geometric fractional distance function $\mathcal{D}_{AB}$ is defined as

$$\mathcal{D}_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{AX}{AB} & \text{for} \quad X \neq A, \; X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

The quotient of two real line segments is defined as a real number.

Remark 3.1.2 The domain of $\mathcal{D}_{AB}(AX)$ is defined as subsets of real line segments. This allows $AX = AA$ which would be excluded from a domain of real line segments because AA does not have two different endpoints.

Theorem 3.1.3 *For any point $X \in AB$, the bijective geometric fractional distance function $\mathcal{D}_{AB}(AX) : AB \to R$ has range $R = [0,1]$.*

Proof. Assume $\mathcal{D}_{AB}(AX) < 0$. Then one of the lengths in the fraction must be negative and we invoke a contradiction with the length of a line segment defined as a positive number (Definition 2.1.2.) If $\mathcal{D}_{AB}(AX) > 1$, then $AX > AB$ and we invoke a contradiction by the implication $AX \not\subseteq AB$. We have excluded from R all numbers less than zero and greater than one. Since $\mathcal{D}_{AB}(AX)$ is a continuous function taking the values zero and one at the endpoints of its domain, the intermediate value theorem requires that the range of $\mathcal{D}_{AB}(AX) : AB \to R$ is $R = [0,1]$. ∎

Corollary 3.1.4 *All line segments have at least one midpoint.*

Proof. (Reproof of Theorem 2.3.21.) $\mathcal{D}_{AB}(AX)$ is a continuous function on the domain AB taking finite values zero and one at the endpoints of its domain. By the intermediate value theorem, there exists a point C in the domain AB for which $\mathcal{D}_{AB}(AC) = 0.5$. By Definition 2.3.18, C is a midpoint of AB. ∎

Theorem 3.1.5 *Every midpoint of a line segment AB is an interior point of AB.*

Proof. If $X \in AB$ is not an interior point of AB, then $X = A$ or $X = B$. In each case respectively, the geometric fractional distance function returns

$$\mathcal{D}_{AB}(AA) = 0 \quad , \quad \text{or} \qquad \mathcal{D}_{AB}(AB) = 1 \quad .$$

A point C is a midpoint of AB if and only if

$$\mathcal{D}_{AB}(AC) = 0.5 \quad .$$

No midpoint can be an endpoint.

Remark 3.1.6 Given the geometric fractional distance function, it is not clear how to compute $\mathcal{D}_{AB}(AX)$ when X is an arbitrary interior point. By Definition 3.1.1, we know that the fraction $\frac{AX}{AB}$ is a real number but we have not yet developed any tools for finding the numerical value. The quotient notation required for computing fractional distance calls for an algebraic notion of distance.

Definition 3.1.7 $\mathcal{D}^{\dagger}_{AB}$ is the algebraic fractional distance function. It is an algebraic expression which totally replicates the behavior of the geometric fractional distance function $\mathcal{D}_{AB}$ on an arbitrary line segment $AB \equiv [a, b]$, *and* it has the added property that its numerical output is easily simplified. The algebraic fractional distance function $\mathcal{D}^{\dagger}_{AB}$ is constrained to be such that

$$\mathcal{D}^{\dagger}_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

for every point $X \in AB$.

Remark 3.1.8 In Definitions 3.1.9 and 3.1.11, we will define two kinds of algebraic fractional distance functions (FDFs.) The purpose in defining two kinds of FDFs will be so that we may compare their properties and then choose the one that exactly replicates the behavior of the geometric FDF $\mathcal{D}_{AB}$.

Definition 3.1.9 The algebraic FDF of the first kind

$$\mathcal{D}'_{AB}(AX) : AB \to [0, 1] \quad ,$$

is a map on subsets of real line segments

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} & \text{for} \quad X \neq A, \; X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad ,$$

where

$$\frac{\|AX\|}{\|AB\|} = \frac{\text{len}[a, x]}{\text{len}[a, b]} \quad ,$$

and $[a, x]$ and $[a, b]$ are the line segments AX and AB expressed in interval notation.

Definition 3.1.10 The norm $\|AX\| = \text{len}[a, x]$ which appears in $\mathcal{D}'_{AB}(AX)$ is defined so that

$$\mathcal{D}'_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

Specifically, the length function is defined as the Euclidean distance between the endpoints of the algebraic representation. Per Definition 2.1.2, we have

$$\text{len}[a, b] = d(a, b) = \big|b - a\big| \quad .$$

Definition 3.1.11 An algebraic fractional distance function of the second kind

$$\mathcal{D}''_{AB}(AX) : [a, b] \to [0, 1] \quad ,$$

is a map on intervals of the form

$$\mathcal{D}''_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\text{len}[a, x]}{\text{len}[a, b]} & \text{for} \quad X \neq A, \; X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Remark 3.1.12 Take note of the main difference between the two algebraic FDFs. The first kind has a geometric domain

$$\mathcal{D}'_{AB}(AX) : AB \to \mathbb{R} \quad ,$$

but the second kind has an algebraic domain

$$\mathcal{D}''_{AB}(AX) : [a, b] \to \mathbb{R} \quad .$$

As a matter of consistency of notation, we have written $\mathcal{D}''_{AB}(AX)$ even when the notation $\mathcal{D}''_{AB}([a, x])$ might better illustrate that the domain of $\mathcal{D}''_{AB}$ is intervals rather than line segments. The reader is so advised.

Axiom 3.1.13 The ordering of $\mathbb{R}$ is such that for any $x, y \in \mathbb{R}$, if

$$x \in [x_1, x_2] = \mathscr{X} \equiv X \quad , \quad \text{and} \qquad y \in [y_1, y_2] = \mathscr{Y} \equiv Y \quad ,$$

then

$$\mathcal{D}_{AB}(AX) > \mathcal{D}_{AB}(AY) \quad \implies \quad x > y \quad .$$

Theorem 3.1.14 *The geometric fractional distance function $\mathcal{D}_{AB}$ is injective (one-to-one) on all real line segments.*

Proof. By Definition 3.1.1, the geometric FDF is

$$\mathcal{D}_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{AX}{AB} & \text{for} \quad X \neq A,\ X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

For proof by contradiction, assume $\mathcal{D}_{AB}$ is not always injective. Then there exists some $X_1 \neq X_2$ such that

$$\frac{AX_1}{AB} = \frac{AX_2}{AB} \quad .$$

The range of $\mathcal{D}_{AB}$ is $[0, 1]$ and it is known that all such $0 \leq x \leq 1$ have an additive inverse element. This allows us to write

$$0 = \frac{AX_2}{AB} - \frac{AX_1}{AB} = \frac{AX_2 - AX_1}{AB} \qquad \Longleftrightarrow \qquad AX_2 = AX_1 \quad .$$

Axiom 2.3.6 gives $AX = AY$ if and only if $X = Y$ so the implication $X_1 = X_2$ contradicts the assumed condition $X_1 \neq X_2$. The geometric fractional distance function $\mathcal{D}_{AB}(AX)$ is injective on all real line segments.

Remark 3.1.15 In Theorem 3.1.14, we have not considered specifically the case in which AB is a line segment of infinite length. There are many numbers x_1 and x_2 such that zero being equal to their difference divided by infinity does not imply that $x_1 = x_2$, *e.g.*:

$$0 = \frac{5 - 3}{\infty} \qquad \not\Longleftrightarrow \qquad 5 = 3 \quad . \tag{3.1}$$

However, $\mathcal{D}_{AB}(AX)$ does not have numbers in its domain. The fraction in Equation (3.1) can never appear when computing $\frac{AX}{AB}$ because $\mathcal{D}_{AB}(AX)$ takes line segments or simply the point A (written as AA in abused line segment notation.)

To be clear, simplifying the expression $\mathcal{D}_{AB}(AX)$ in the general case requires some supplemental constraint like $AB = cAX$ for some scalar c. With a such a constraint, and by way of Axiom 2.3.7, we may evaluate the quotient as

$$\frac{AX}{AB} = \frac{cAB}{AB} = c \quad .$$

Without such auxiliary constraints, we have no general method for the evaluation of the quotient. Theorem 3.1.14 holds, however, because numbers such as the ∞ in the denominator of Equation (3.1) will be used only to compute $\mathcal{D}^{\dagger}_{AB}(AX)$ when we introduce the norm $\|AX\|$. The main feature distinguishing the algebraic FDF $\mathcal{D}^{\dagger}_{AB}$ from the geometric FDF $\mathcal{D}_{AB}$ is that the former allows us to compute the quotient in the general case with no requisite auxiliary constraints. Therefore, we might write $\mathcal{D}^{\dagger}_{AB}(AX;x)$ to show that is is a function of AX *and* a chart x on AB, or $\mathcal{D}^{\dagger}_{AB}([a,x];x)$ as mentioned earlier, but we will not write that explicitly. In the absence of words to the contrary and if AB is a real line segment, then it should be assumed that the chart is the standard Euclidean coordinate.

Theorem 3.1.16 *The geometric fractional distance function $\mathcal{D}_{AB}$ is surjective (onto) on all real line segments.*

Proof. Given the range $R=[0,1]$ proven in Theorem 3.1.3, proof follows from the notion of geometric fractional distance.

Remark 3.1.17 Now that we have shown a few of the elementary properties of the geometric FDF, we will continue to do so and also examine the similar behaviors of the algebraic FDFs of the first and second kinds.

Conjecture 3.1.18 The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ is injective (one-to-one) on all real line segments. (This is proven in Theorem 6.1.4.)

Theorem 3.1.19 *The algebraic fractional distance function of the second kind $\mathcal{D}''_{AB}$ is not injective (one-to-one) on all real line segments.*

Proof. Recall that Definition 3.1.11 gives $\mathcal{D}''_{AB}:[a,b]\to[0,1]$ as

$$\mathcal{D}''_{AB}(AX)=\begin{cases} 1 & \text{for} \quad X=B \\ \dfrac{\text{len}[a,x]}{\text{len}[a,b]} & \text{for} \quad X\neq A,\ X\neq B \\ 0 & \text{for} \quad X=A \end{cases} \quad .$$

Injectivity requires that

$$\mathcal{D}''_{AB}(AX)=\mathcal{D}''_{AB}(AY) \quad \iff \quad [a,x]=[a,y] \quad \iff \quad x=y \quad .$$

Let $n,m\in\mathbb{N}$ be such that $n\neq m$, and also such that $n\in\mathscr{N}\equiv N$ and $m\in\mathscr{M}\equiv M$. We have

$$\mathcal{D}''_{\mathbf{AB}}(AN)=\frac{\text{len}[0,n]}{\text{len}[0,\infty]}=0 \quad , \quad \text{and} \qquad \mathcal{D}''_{\mathbf{AB}}(AM)=\frac{\text{len}[0,m]}{\text{len}[0,\infty]}=0 \quad .$$

Therefore, the algebraic FDF of the second kind is not injective on all real line segments because

$$\mathcal{D}''_{AB}(AN)=\mathcal{D}''_{AB}(AM) \quad \not\Longleftrightarrow \quad n=m \quad .$$

Remark 3.1.20 At this point, we can rule out $\mathcal{D}''_{AB}$ as the definition of $\mathcal{D}^{\dagger}_{AB}$ because the geometric FDF $\mathcal{D}_{AB}$ which constrains $\mathcal{D}^{\dagger}_{AB}$ is one-to-one. If $\mathcal{D}_{AB}$ is one-to-one on all real line segments, then so is $\mathcal{D}^{\dagger}_{AB}$.

Carefully note that the domain of the algebraic FDF of the first kind is line segments rather than algebraic intervals. We have

$$\mathcal{D}'_{AB}(AX) : AB \to [0,1] \quad , \quad \text{and} \qquad \mathcal{D}''_{AB}(AX) : [a,b] \to [0,1] \quad .$$

Taking for granted that we will prove the injectivity of $\mathcal{D}'_{AB}$ in Theorem 6.1.4, this distinction of domain—AB versus $[a,b]$—will prohibit the breakdown in the one-to-one property when a point $X \in AB$ can have many different numbers in its algebraic representation. An assumption that the domain of the algebraic FDF is an algebraic interval $[a,b]$ is likely a root cause of ***much pathology in modern analysis***.

Theorem 3.1.21 *The geometric fractional distance function $\mathcal{D}_{AB}$ is continuous everywhere on the domain* $\mathbf{AB}$.

Proof. To prove that $\mathcal{D}_{AB}$ is continuous on $\mathbf{AB} \equiv [0,\infty]$, it will suffice to show that $\mathcal{D}_{\mathbf{AB}}$ is continuous at the endpoints and an interior point.

• (*Interior point*) A function $f(x)$ is continuous at an interior point x_0 of its domain $[a,b]$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0) \quad .$$

In terms of the geometric FDF, the statement that $\mathcal{D}_{\mathbf{AB}}$ is continuous at an interior point $X_0 \in \mathbf{AB}$ becomes

$$\lim_{X \to X_0} \mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{D}_{\mathbf{AB}}(AX_0) \quad .$$

Obviously, $\mathcal{D}_{\mathbf{AB}}$ satisfies the definition of continuity on the interior of $\mathbf{AB}$.

• (*Endpoint A*) A function $f(x)$ is continuous at the endpoint a of its domain $[a,b]$ if and only if

$$\lim_{x \to a^+} f(x) = f(a) \quad .$$

We conform to this definition of continuity with

$$\lim_{X \to A^+} \mathcal{D}_{\mathbf{AB}}(AX) = \lim_{X \to A^+} \frac{AX}{\mathbf{AB}} = \frac{AA}{\mathbf{AB}} = \mathcal{D}_{\mathbf{AB}}(AA) \quad .$$

• (*Endpoint B*) A function $f(x)$ is continuous at the endpoint b of its domain $[a,b]$ if and only if

$$\lim_{x \to b^-} f(x) = f(b) \quad .$$

We conform to this definition with

$$\lim_{X\to B^-} \mathcal{D}_{\mathbf{AB}}(AX) = \lim_{X\to B^-} \frac{AX}{\mathbf{AB}} = \frac{\mathbf{AB}}{\mathbf{AB}} = \mathcal{D}_{\mathbf{AB}}(\mathbf{AB}) \quad .$$

The geometric FDF is continuous everywhere on its domain.

Theorem 3.1.22 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ is not continuous everywhere on the domain* $\mathbf{AB}$.

Proof. A function $f(x)$ with domain $x \in [a,b]$ is continuous at b if

$$\lim_{x\to b^-} f(x) = f(b) \quad ,$$

In terms of $\mathcal{D}'_{AB}$, the statement that $\mathcal{D}'_{\mathbf{AB}}$ is continuous at B becomes

$$\lim_{X\to B} \mathcal{D}'_{\mathbf{AB}}(AX) = \mathcal{D}'_{\mathbf{AB}}(\mathbf{AB}) = 1 \quad .$$

Evaluation yields

$$\lim_{X\to B} \mathcal{D}'_{\mathbf{AB}}(AX) = \lim_{x\to\infty} \frac{\text{len}[0,x]}{\text{len}[0,\infty]} = \lim_{x\to\infty} x\frac{1}{\infty} = \lim_{x\to\infty} 0 \neq 1 = \mathcal{D}'_{\mathbf{AB}}(AB) \quad .$$

The algebraic FDF of the first kind is not continuous everywhere on all real line segments.

Remark 3.1.23 In Theorem 3.1.22, we have shown that the limit approaches zero rather than the unit value required for $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AB)$ to agree with $\mathcal{D}_{\mathbf{AB}}(AB)$. However, we may also write this limit as

$$\lim_{x\to\infty} \frac{1}{\infty}x = \lim_{\substack{x\to\infty \\ y\to\infty}} \frac{x}{y} = \lim_{y\to\infty} \infty\frac{1}{y} = \frac{\infty}{\infty} = \text{undefined} \quad .$$

Perhaps, then, it would be better to write simply

$$\lim_{x\to\infty} \frac{x}{\infty} = \frac{\infty}{\infty} = \text{undefined} \neq 1 \quad .$$

In any case, we have shown that an elementary evaluation does not produce the correct limit at infinity. Therefore, we should also examine the Cauchy definition of the limit relying on the ε–δ formalism.

Theorem 3.1.24 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ does not converge to a Cauchy limit at infinity.*

Proof. According to the Cauchy definition of the limit of $f : D \to R$ at infinity, we say that

$$\lim_{x\to\infty} f(x) = l \quad ,$$

if and only if

$$\forall\varepsilon > 0 \quad \exists\delta > 0 \quad \text{s.t} \quad \forall x \in D \quad ,$$

we have

$$0 < |x - \infty| < \delta \qquad \Longrightarrow \qquad |f(x) - l| < \varepsilon \quad .$$

There is no $\delta > \infty$ so $\mathcal{D}'_{AB}(AX)$ fails the Cauchy criterion for convergence to a limit at infinity.

Remark 3.1.25 In general, the above Cauchy definition of a limit fails for any limit at infinity because there is never a δ greater than infinity. Usually this issue is worked around with the metric space definition of a limit at infinity but it is a ***main result of this analysis*** that we will develop a technique for taking a limit at infinity with the normal Cauchy prescription. This result appears in Section 6.1.

Remark 3.1.26 The algebraic FDF $\mathcal{D}^{\dagger}_{AB}$ exists by definition. It is a function which has every behavior of the geometric FDF $\mathcal{D}_{AB}$ and also adds the ability to compute numerical ratios between the lengths of any two real line segments. Numbers being generally within the domain of algebra, the geometric FDF returns a fraction that we have no general way to simplify. Since it is hard to conceive of an irreducible analytical form for the algebraic FDF other than $\mathcal{D}'_{AB}$ and $\mathcal{D}''_{AB}$, it is somewhat paradoxical that neither of them replicate the global behavior of the algebraic FDF $\mathcal{D}^{\dagger}_{AB}$. After developing some more material, we will show in Section 6.1 that $\mathcal{D}^{\dagger}_{AB}$ is $\mathcal{D}'_{AB}$ after all. We will prevent an unwarranted assumption about infinity from sneakily propagating into the present analysis. Then we will fix the discontinuity of $\mathcal{D}'_{AB}$ which we have demonstrated in Theorems 3.1.22 and 3.1.24.

Theorem 3.1.27 *If x is a real number in the algebraic representations of both $X \in AB$ and $Y \in AB$, then $X = Y$.*

Proof. If $X \neq Y$, then

$$\mathcal{D}^{\dagger}_{AB}(AX) \neq \mathcal{D}^{\dagger}_{AB}(AY) \quad .$$

If $x \in X$ and $x \in Y$, then it is possible to make cuts at X and Y such that

$$\mathcal{D}^{\dagger}_{AB}(AX) = \frac{\text{len}[a,x]}{\text{len}[a,b]} = \mathcal{D}^{\dagger}_{AB}(AY) \quad .$$

This contradiction requires $X = Y$.

§3.2 Comparison of Real and Natural Numbers

The main result of this section is to prove via analysis of FDFs that there exist real numbers greater than any natural number. Consequently, $\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0$ cannot be the empty set.

Definition 3.2.1 Every interval has a number at its center. The number at the center of an interval $[a,b]$ is defined as the average of a and b if an average can be computed. Otherwise, the number in the center is the unique number c such that for every $c+x$ in the interval, there is a corresponding $c-x$ in it. This holds for all intervals $[a,b)$, $(a,b]$, and (a,b).

Theorem 3.2.2 *There exists a unique real number halfway between zero and infinity.*

Proof. By Theorem 2.3.23 and by Definition 2.3.18, there exists one midpoint C of every line segment AB such that

$$\mathcal{D}_{AB}(AC) = 0.5 \quad .$$

Recalling that we have defined $\mathcal{D}_{AB}(AX) = \mathcal{D}^{\dagger}_{AB}(AX)$ for all $X \in AB$, and recalling that $\mathbf{AB} \equiv [0,\infty]$, it follows that

$$\mathcal{D}^{\dagger}_{\mathbf{AB}}(AC) = 0.5 \quad .$$

Using $C \equiv \mathscr{C} = [c_1, c_2]$, Axiom 2.3.10 and Definition 2.3.15 require

$$\mathbf{AB} = AC + CB \qquad \Longleftrightarrow \qquad [0,\infty] = [0,c_1) \cup \mathscr{C} \cup (c_2,\infty] \quad .$$

It follows that

$$\mathscr{C} \subset \mathbb{R} \quad .$$

Every possible number that can be in the algebraic representation of the point C is a real number. If $c_1 = c_2 = c$, then $c \in \mathbb{R}$ is the unique real number halfway between zero and infinity. If $c_1 \neq c_2$, then, by Definition 3.2.1, the number at the center of $[c_1, c_2]$ is the unique real number halfway between zero and infinity.

Remark 3.2.3 How can $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AC) = 0.5$ when Definition 3.1.9 gives

$$\mathcal{D}'_{\mathbf{AB}}(AC) = \frac{\text{len}[0,c]}{\infty} \quad ?$$

The prevailing assumption about infinity is

$$x \in \mathbb{R} \qquad \Longleftrightarrow \qquad \frac{x}{\infty} = 0 \quad . \tag{3.2}$$

If Equation (3.2) is true, then either (i) there exists a line segment without a midpoint, or (ii) the geometric and algebraic fractional distance functions do not agree for every X in an arbitrary AB.

Every line segment does have a midpoint (Theorem 2.3.23) and our fractional distance functions are defined to always agree (Definition 3.1.7.) Therefore, Equation (3.2), which is a statement dependent on the assumed properties of ∞, must be reformulated. In Section 4.1, we will define notation for subsets of $\mathbb{R}$ consisting of all numbers having fractional distance $\mathcal{X}$ with respect to **AB** meaning that $\frac{x}{\infty} = \mathcal{X}$. The sets will be labeled $\mathbb{R}_{\aleph}^{\mathcal{X}}$ most generally with $0 < \mathcal{X} < 1$ but it will follow that $\mathbb{R}_{\aleph}^{0}$ is the set of all real numbers having zero fractional distance with respect to **AB**. We know that $\mathbb{R}_0 \subset \mathbb{R}_{\aleph}^{0}$ but it shall remain to be determined whether or not there are real numbers greater than any natural number yet still having zero fractional distance with respect to **AB**. In Section 7.4, we will closely examine whether or not such numbers ought to exist.

While we will postpone the definition of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ to Section 4.1, and while the formal construction of $\mathbb{R}_{\aleph}^{\mathcal{X}}$ by equivalence classes of Cauchy sequences will not appear until Section 4.2, here we will go ahead and answer the question, "How can $\mathcal{D}_{\mathbf{AB}}^{\dagger}(AC) = 0.5$ when Definition 3.1.9 gives

$$\mathcal{D}'_{\mathbf{AB}}(AC) = \frac{\text{len}[0,c]}{\infty} \quad ?\text{"}$$

The answer is that Equation (3.2) must be reformulated as

$$x \in \mathbb{R}_{\aleph}^{0} \qquad \Longleftrightarrow \qquad \frac{x}{\infty} = 0 \quad ,$$

if we are to avoid harsh contradictions in the definitions of our FDFs. Regarding Theorem 3.2.2 and the present question which follows, the real numbers in the algebraic representation of the geometric midpoint of **AB** shall be

$$x \in \mathbb{R}_{\aleph}^{0.5} \qquad \Longleftrightarrow \qquad \frac{x}{\infty} = 0.5 \quad .$$

In addition to motivating the soon-to-be-defined $\mathbb{R}_{\aleph}^{\mathcal{X}}$ notation, the present remark illustrates the reasoning behind allowing geometric points to be represented as entire intervals $X \equiv \mathscr{X}$. The reason is that many real numbers divided by infinity give zero but only the geometric left endpoint of **AB** will have vanishing fractional distance. For instance, if $x \in \mathbb{R}_{\aleph}^{0.5}$ and n is a natural number having zero fractional magnitude with respect to infinity, then

$$\frac{x+n}{\infty} = \frac{x}{\infty} + \frac{n}{\infty} = 0.5 + 0 \quad .$$

Obviously, $x \in \mathbb{R}_{\aleph}^{0.5}$ is not a unique number though the midpoint C is a unique point.

Definition 3.2.4 If $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is the set of all numbers whose fractional distance with respect to **AB** is $\mathcal{X}$, and if $0 < \mathcal{X} < 1$, then $\aleph_{\mathcal{X}}$ is the number in the center of the interval $\mathbb{R}_{\aleph}^{\mathcal{X}} = (a, b)$ in the sense that for every $\aleph_{\mathcal{X}} + n \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ there exists an $\aleph_{\mathcal{X}} - n \in \mathbb{R}_{\aleph}^{\mathcal{X}}$.

Remark 3.2.5 The reader is invited to recall that Euler often employed the letter i to refer to an infinitely large integer. Euler made use of the number $\frac{i}{2}$ for proofs in his most seminal works [5–7]. Therefore, we are certainly introducing nothing new with the $\aleph_{\mathcal{X}}$ notation because $\frac{i}{2} \sim \aleph_{0.5}$.

Main Theorem 3.2.6 *Some elements of* $\mathbb{R}$ *are greater than every element of* $\mathbb{N}$.

Proof. Let $\mathbf{AB}$ have a midpoint C so that $\mathcal{D}_{\mathbf{AB}}(AC) = 0.5$. Then every real number $c \in [c_1, c_2] \equiv C$ is greater than any $n \in \mathbb{N}$ because $\frac{n}{\infty} = 0$ implies $n \in \mathscr{A} \equiv A$ through the definition $\mathcal{D}_{\mathbf{AB}}(AA) = 0$. $\mathcal{D}_{\mathbf{AB}}$ is one-to-one so by Axiom 3.1.13 giving for $x \in X$ and $y \in Y$

$$\mathcal{D}_{AB}(AX) > \mathcal{D}_{AB}(AY) \quad \Longrightarrow \quad x > y \ \ ,$$

we find that every $c \in \mathscr{C} \subset \mathbb{R}$ is greater than every $n \in \mathbb{N}$. Generally, every $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ is greater than any natural number whenever $\mathcal{X} > 0$.

Corollary 3.2.7 $\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0$ *is not the empty set.*

Proof. Definition 2.1.9 defines $\mathbb{R}_0$ as the subset of $\mathbb{R}$ whose elements are less than some element of $\mathbb{N}$. We have proven in Main Theorem 3.2.6 that some elements of $\mathbb{R}$ are not in $\mathbb{R}_0$. It follows that

$$\mathbb{R}_\infty \neq \varnothing \ \ , \quad \text{because} \quad \mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0 \ \ .$$

§3.3 Comparison of Cuts in Lines and Points in Line Segments

In this section, we will make clarifications regarding the cases in which an interior point of a line segment can or cannot be identified with a unique real number. Namely, we distinguish cases in which $X \equiv x$ and $X \equiv \mathscr{X} = [x_1, x_2]$ with $x_1 \neq x_2$.

Theorem 3.3.1 *If* AB *is a real line segment of finite length* $L \in \mathbb{R}_0$*, then every point* $X \in AB$ *has a unique algebraic representation as one and only one real number.*

Proof. Let $a, b \in \mathbb{R}_0$ and $AB \equiv [a, b]$. The algebraic FDF $\mathcal{D}_{AB}^\dagger$ is defined to behave exactly as the geometric FDF $\mathcal{D}_{AB}$. Therefore, $\mathcal{D}_{AB}^\dagger$ must be one-to-one (injective.) By Definition 2.3.15, every point in a real line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \ \ .$$

Therefore, the present theorem will be proven if we show that $x_1 = x_2$ for all $X \in AB$ with $L \in \mathbb{R}_0$. To initiate proof by contradiction, assume $x_1, x_2 \in \mathbb{R}_0$ and $x_1 \neq x_2$. (The validity of this condition follows from $L \in \mathbb{R}_0$.) Then

$$\min[\mathcal{D}^{\dagger}_{AB}(AX)] = \frac{\text{len}[a, x_1]}{\text{len}[a, b]} = \frac{x_1 - a}{b - a} \quad ,$$

and

$$\max[\mathcal{D}^{\dagger}_{AB}(AX)] = \frac{\text{len}[a, x_2]}{\text{len}[a, b]} = \frac{x_2 - a}{b - a} \quad .$$

The one-to-one property of $\mathcal{D}^{\dagger}_{AB}$ requires that

$$\frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a} \qquad \Longleftrightarrow \qquad x_1 = x_2 \quad .$$

This contradicts the assumption $x_1 \neq x_2$. The theorem is proven.

Theorem 3.3.2 *If AB is a real line segment of infinite length $L = \infty$, then no point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. By Definition 2.3.15, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

It follows that

$$\min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \infty]} = \frac{x_1}{\infty} \quad ,$$

Now suppose that $x_0 \in \mathbb{R}^+_0$ where the superscript "+" indicates the positive-definite subset. Further suppose $z = x_1 + x_0$ so that $z > x_1$. Then

$$\frac{\text{len}[0, z]}{\text{len}[0, \infty]} = \frac{z}{\infty} = \frac{x_1 + x_0}{\infty} = \frac{x_1}{\infty} = \min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] \quad .$$

Invoking the single-valuedness of bijective functions, we find that

$$\min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \max[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{x_2}{\infty} \qquad \Longrightarrow \qquad x_1 < z \leq x_2 \quad .$$

Therefore $x_1 \neq x_2$ and the theorem is proven.

Example 3.3.3 This example illustrates some of the underlying machinations associated with the many-to-one relationship between numbers and points in an infinitely long line segment. If we separate an endpoint from a closed algebraic interval, then we may write

$$[a, b] = \{a\} \cup (a, b] \quad .$$

To separate an endpoint from a line segment we write

$$AB = A + AB \quad .$$

If A has an algebraic representation $\mathscr{A}$ such that $\text{len}(\mathscr{A}) > 0$, then the only way that we can leave the length of AB unchanged after removing A is for AB to have infinite length. Given $\text{len}(\mathscr{A}) > 0$, observe that

$$\|AB\| - \text{len}\,\mathscr{A} = \|AB\| \quad \Longleftrightarrow \quad \|AB\| = \infty \quad .$$

Remark 3.3.4 Theorems 3.3.1 and 3.3.2 do not cover all cases of $\text{len}(AB) = L$. For instance, four coarse bins of L are

- $L \in \mathbb{R}_0$
- $L \in \mathbb{R}_\aleph^0 \setminus \mathbb{R}_0$ (L larger than any $n \in \mathbb{N}$ yet not so large that $\frac{L}{\infty} > 0$.)
- $L \in \mathbb{R}_\infty \setminus \mathbb{R}_\aleph^0$ (which is also written $L \in \mathbb{R}_\aleph^{\mathcal{X}} \cup \mathbb{R}_\aleph^1$ when $\mathbb{R}_\aleph^{\mathcal{X}}$ is understood to be $0 < \mathcal{X} < 1$, as in Section 4.1)
- $L = \infty$.

We have not considered the two intermediate cases of finite L. The lesser case is finite $L \in \mathbb{R}_\aleph^0 \setminus \mathbb{R}_0$. Since we have not yet introduced numbers through which to describe the lesser case, and we will not decide $\mathbb{R}_\aleph^0 \setminus \mathbb{R}_0 = \varnothing$ until Section 7.4, we cannot at this time prove the result regarding the multi-valuedness of points in line segments having $L \in \mathbb{R}_\aleph^0 \setminus \mathbb{R}_0$. The limit of the third case as $L \in \mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_0^1$ is proven to be many-to-one in Theorem 6.2.1.

§4 The Neighborhood of Infinity

§4.1 Intermediate Neighborhoods of Infinity

In this section, we will develop notation useful for describing real numbers whose fractional magnitude with respect to infinity is greater than zero.

Definition 4.1.1 The number $\aleph_{\mathcal{X}}$ is defined to have the property

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \mathcal{X} \quad .$$

Equivalently, if $\aleph_{\mathcal{X}} \in \mathscr{X} \equiv X \in \mathbf{AB}$, then

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \quad .$$

Remark 4.1.2 We have shown in Theorem 3.3.2 that there are many real numbers in the algebraic representation of $X \in \mathbf{AB}$. When X is not an endpoint of $\mathbf{AB}$, $\aleph_{\mathcal{X}}$ can be thought of the as the number in the center of the interval $(x_1, x_2) = \mathscr{X} \equiv X$. Definition 3.2.1 defines the number in the center of $\mathscr{X}$ as the average of x_1 and x_2 if the average is computable, but here we have no way to determine the least and greatest numbers in the algebraic representation of X. It is useful, therefore, to think of $\aleph_{\mathcal{X}}$ as the number in the center of $\mathscr{X}$ in the sense that for every $\aleph_{\mathcal{X}} + |b| \in \mathscr{X}$ there exists a $\aleph_{\mathcal{X}} - |b| \in \mathscr{X}$. For the special cases of $\aleph_0$ and $\aleph_1$, we should not think of them as being in the centers of the intervals $\mathscr{A} \equiv A$ and $\mathscr{B} \equiv B$. Instead, $\aleph_0$ is the least number in $\mathscr{A} \equiv A \in \mathbf{AB}$ and $\aleph_1$ is the greatest number in $\mathscr{B} \equiv B \in \mathbf{AB}$.

Definition 4.1.3 For $0 < \mathcal{X} < 1$, $\mathbb{R}_\aleph^\mathcal{X}$ is a subset of positive real numbers $\mathbb{R}^+$ such that

$$\mathbb{R}_\aleph^\mathcal{X} = \left\{ \aleph_\mathcal{X} + b \;\middle|\; |b| \in A \in \mathbf{AB},\ \mathcal{D}_{\mathbf{AB}}(AA) = 0 \right\} \quad .$$

The set $\mathbb{R}_\aleph^\mathcal{X}$ is called the whole neighborhood of $\aleph_\mathcal{X}$. The set $\{\mathbb{R}_\aleph^\mathcal{X}\}$ of all $\mathbb{R}_\aleph^\mathcal{X}$, meaning the union of $\mathbb{R}_\aleph^\mathcal{X}$ for every $0 < \mathcal{X} < 1$, is called the set of all intermediate neighborhoods of $\mathbb{R}$. We will also call $\mathbb{R}_\aleph^\mathcal{X}$ the neighborhood of numbers that are $100 \times \mathcal{X}\%$ of the way down the real number line. (These conventions ignore the negative branch of $\mathbb{R}$.)

Definition 4.1.4 It will also be useful to define a set $\mathbb{R}_0^\mathcal{X} \subseteq \mathbb{R}_\aleph^\mathcal{X}$ such that $0 < \mathcal{X} < 1$ and

$$\mathbb{R}_0^\mathcal{X} = \left\{ \aleph_\mathcal{X} + b \;\middle|\; b \in \mathbb{R}_0 \right\} \quad .$$

The set $\mathbb{R}_0^\mathcal{X}$ is called the natural neighborhood of $\aleph_\mathcal{X}$ because here we have constrained b to be less than some $n \in \mathbb{N}$. $\{\mathbb{R}_0^\mathcal{X}\}$ is the union of $\mathbb{R}_0^\mathcal{X}$ for every $0 < \mathcal{X} < 1$.

Definition 4.1.5 Every number of the form $x = \aleph_\mathcal{X} + b$ has a big part $\aleph_\mathcal{X}$ and a little part b. It is understood that $b < \aleph_\mathcal{X}$ for any $\mathcal{X} > 0$. We define notations

$$\mathrm{Big}(\aleph_\mathcal{X} + b) = \aleph_\mathcal{X} \quad , \quad \text{and} \quad \mathrm{Lit}(\aleph_\mathcal{X} + b) = b \quad .$$

Remark 4.1.6 We have omitted from Definitions 4.1.3 and 4.1.4 the cases of $\mathcal{X} = 0$ and $\mathcal{X} = 1$ though they do follow more or less directly. The main issue is that we must restrict the sign of b to keep the elements of the set within the totally real interval $[0, \infty) \subset \mathbb{R}$. For $\mathcal{X} = 0$, the little part b is non-negative and for $\mathcal{X} = 1$ it is negative-definite.

The difference between the natural neighborhoods $\mathbb{R}_0^\mathcal{X}$ and the whole neighborhoods $\mathbb{R}_\aleph^\mathcal{X}$ is that b is not restricted to $\mathbb{R}_0$ in the latter. In Definition 4.1.4, we did not give the condition on b in terms of the absolute value as in Definition 4.1.3 because $\mathbb{R}_0$ contains negative numbers while $b \in \mathbf{AB} \equiv [0, \infty]$ is strictly non-negative. The main purpose in defining distinct sets $\{\mathbb{R}_0^\mathcal{X}\}$ and $\{\mathbb{R}_\aleph^\mathcal{X}\}$ is this: we know that there exist numbers larger than any $b \in \mathbb{R}_0$ (Main Theorem 3.2.6) but we do not know whether or not all such numbers have greater than zero fractional magnitude with respect to $\mathbf{AB}$. We will revisit this issue in Section 7.4. In the meantime, we will be careful to treat $\mathbb{R}_0^{\mathcal{X}}$ and $\mathbb{R}_\aleph^{\mathcal{X}}$ as distinct sets which may or may not be equal.

Definition 4.1.7 The whole neighborhood of the origin is

$$\mathbb{R}_\aleph^0 = \left\{ x \mid x \in \mathscr{A} \equiv A \in \mathbf{AB} \right\} \quad ,$$

and the natural neighborhood of the origin is

$$\mathbb{R}_0^0 = \left\{ x \mid x \in \mathbb{R}_0,\ x \geq 0 \right\} \quad ,$$

Remark 4.1.8 Note that $\mathbb{R}_0 \not\subseteq \mathbb{R}_0^0 \subseteq \mathbb{R}_\aleph^0$ because $\mathbb{R}_0$ contains positive and negative numbers, as per Definition 2.1.9.

Definition 4.1.9 A real number x is said to be in the neighborhood of the origin if and only if

$$x \in X \quad , \qquad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AX) = 0 \quad .$$

All such numbers are said to be $x \in \mathbb{R}_\aleph^0$. Every real number not in the neighborhood of the origin is said to in the neighborhood of infinity. A positive real number x is said to be in the neighborhood of infinity if and only if

$$x \in X \quad , \qquad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AX) \neq 0 \quad .$$

Remark 4.1.10 Definition 2.1.10 states that $\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0$. Therefore, if $\mathbb{R}_\aleph^0 \setminus \mathbb{R}_0^0 \neq \varnothing$, meaning that there do exist real numbers greater than any natural number yet not great enough to have non-zero fractional distance with respect to **AB**, then the set $\mathbb{R}_\infty$ will contain numbers in the neighborhood of the origin *and* numbers in the neighborhood of infinity. To avoid ambiguity, we will not use the symbol $\mathbb{R}_\infty$ and instead we will mostly use the detailed set enumeration scheme given in the present section. With this scheme of distinct whole and natural neighborhoods, we have left room judiciously for numbers in the neighborhood of the origin which are larger than any natural number. In other work [8, 9], we used the semantic convention that every number in the neighborhood of the origin is less than some natural number. That meant $\mathbb{R}_0$ was the set of all real numbers in the neighborhood of the origin. The present convention, however, is better suited to the fuller analysis presently given. The reader should carefully note that the present neighborhood of the origin $\mathbb{R}_\aleph^0$ includes all numbers which have zero fractional distance along the real number line, even if some of those numbers are larger than any $n \in \mathbb{N}$.

Definition 4.1.11 The δ-neighborhood of a number $x \in \mathbb{R}$ is an interval $(x-\delta, x+\delta)$ or some closed or half-open permutation thereof. While there is no inherent constraint on the magnitude of δ, here we will take "δ-neighborhood" to imply $\delta \in \mathbb{R}_0$. We will use the convention that the Ball function defines an open δ-neighborhood as

$$\text{Ball}(x, \delta) = (x - \delta, x + \delta) \quad .$$

Definition 4.1.12 The δ-neighborhood of an interior point $X \in AB$ is a line segment YZ where

$$\big|\mathcal{D}_{AB}(AX) - \mathcal{D}_{AB}(AY)\big| = \big|\mathcal{D}_{AB}(AX) - \mathcal{D}_{AB}(AZ)\big| = \delta \quad .$$

Remark 4.1.13 Without regard to the δ-neighborhood of any point or number, we have defined neighborhoods with the geometric FDF, as in Definition 4.1.9. If $\mathcal{D}_{\mathbf{AB}}(AX) = 0$, then the numbers in the algebraic representation of X are said to be in the neighborhood of the origin. They are said to be in the neighborhood of infinity otherwise. Neither of these neighborhoods, neither that of the origin nor that of infinity, are defined formally as δ-neighborhoods though such a definition may be inferred. In advance of the following definition for $\mathbb{R}_{\aleph}^{\mathcal{X}}$ (Definition 4.1.14), recall that Definition 3.2.4 gave $\aleph_{\mathcal{X}}$ as the number in the center of the interval $\mathbb{R}_{\aleph}^{\mathcal{X}} = (a, b)$.

Definition 4.1.14 An alternative definition for $\mathbb{R}_{\aleph}^{\mathcal{X}}$ valid in the neighborhood of infinity, meaning for $0 < \mathcal{X} < 1$, is

$$\mathbb{R}_{\aleph}^{\mathcal{X}} = \big\{\aleph_{\mathcal{X}} \pm b \;\big|\; b \in \mathbb{R}_{\aleph}^{0}\big\} \quad .$$

This definition is totally equivalent to Definition 4.1.3.

§4.2 Equivalence Classes for Intermediate Natural Neighborhoods of Infinity

Euclid's definition of $\mathbb{R}$ is inherently a geometric one based on the measurement of quantity. The purpose of Cantor's definition by Cauchy equivalence classes [2,10–12] is to give an algebraic definition based on rationals. In this section, we will append the algebraic Cauchy definition to the Euclidean definition given in Section 2.1. This totally algebraic hybrid construction will not unduly exclude the neighborhood of infinity from $\mathbb{R}$. In its ordinary incarnation, the Cauchy definition contradicts the axiom that $\mathbb{R} = (-\infty, \infty)$ because it precludes the existence of numbers larger than any natural number. We have shown that if every number in the interval $(-\infty, \infty)$ is to be a real number, then there must exist numbers such as $\aleph_{0.5}$ which are greater than any natural number. In this section, we will modify the Cauchy definition so that it will support the underlying geometric construction and facilitate the algebraic construction of numbers in the neighborhood of infinity. Here we will only construct the natural neighborhoods because the equality or inequality of $\mathbb{R}_{0}^{\mathcal{X}}$ and $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is not treated until Section 7.4.

Definition 4.2.1 The rational numbers $\mathbb{Q}$ are an Archimedean number field satisfying all of the well-known field axioms given in Section 5.4.

Definition 4.2.2 A sequence $\{x_n\}$ is a Cauchy sequence if and only if

$$\forall\delta\in\mathbb{Q}\quad\exists m,n,N\in\mathbb{N}\quad\text{s.t.}\quad m,n>N\quad\implies\quad\big|x_n-x_m\big|<\delta\ \ .$$

Definition 4.2.3 We say a relation is an equivalence relation if and only if (i) S is a set, (ii) every $x\in S$ is related to x meaning the relation is reflexive, (iii) for every $x,y\in S$ the relation of x to y implies the relation of y to x meaning the relation is symmetric, and (iv) for every $x,y,z\in S$ the relation of x to y and the relation of y to z together imply the relation of x to z meaning the relation is transitive. The equivalence class of $x\in S$, namely the set of all objects which are related to x by an equivalence relation, is denoted $[x]$. At times we will write $[x]=[\{x_n\}]$ or $[x]=[(x_n)]$ to emphasize that the equivalence relation is among Cauchy sequences where $\{x_n\}$ and (x_n) have the same meaning.

Definition 4.2.4 $C_\mathbb{Q}$ is the set of all Cauchy sequences of rational numbers.

Remark 4.2.5 Usually the Cauchy construction of $\mathbb{R}$ is formulated as, "Every $x\in\mathbb{R}$ is some Cauchy equivalence class $[x]\subset C_\mathbb{Q}$," but here we will take a slightly different approach.

Axiom 4.2.6 Every $x\in\mathbb{R}$ may be constructed algebraically as (i) a Cartesian product of Cauchy equivalence classes of rational numbers, or (ii) a partition of all such products.

Axiom 4.2.7 Every $x\in\mathbb{R}_0\subset\mathbb{R}$ is a Cauchy equivalence class of rationals $x=[x]\subset C_\mathbb{Q}$ and also a Dedekind partition of $\mathbb{Q}$ in canonical form $x=(L,R)$. (Dedekind cuts are defined in Section 7.5.)

Remark 4.2.8 Axiom 4.2.7 grants that the reals are constructed by Cauchy equivalence classes or Dedekind partitions (cuts) in the most canonical sense ***if one takes the complementary axiom that every real number is less than some natural number***. We do not take that axiom so we specify $x\in\mathbb{R}_0$ as the object of relevance.

Remark 4.2.9 Cantor's Cauchy construction of $\mathbb{R}$, like the Dedekind construction, is said to be "rigorous" because it begins with the rationals $\mathbb{Q}$. However, before one may assume the existence of $\mathbb{Q}$, one must define zero because $0\in\mathbb{Q}$ but $0\notin\mathbb{N}$. Therefore, to be rigorous, one simply may not assume $\mathbb{Q}$ as a consequence of $\mathbb{N}$. To introduce zero, we will introduce the line segment

AB and define zero as the least number in the algebraic representation of the geometric point A. In other words, $0 = \aleph_0$. It is true that this present approach can be criticized as being "not rigorous" because we have assumed **AB** in the same way that others assumed $\mathbb{Q}$ but the present construction is "more rigorous" because it bumps that which is assumed down to a more primitive level, *i.e.*: Euclid's principles of geometry [1].

Definition 4.2.10 The symbol $\hat{0}$ is an instance of the number zero with the instruction not to do any of zero's absorptive operations. The absorptive operations of zero are

$$0 + x = x \quad , \quad \text{and} \qquad 0 \cdot x = 0 \quad .$$

Expressions containing $\hat{0}$ are not to be simplified by either of these operations.

Axiom 4.2.11 For every Cauchy sequence $\{x_n\}$ in the equivalence class $[x] \subset C_{\mathbb{Q}}$, there exists another Cauchy sequence $\{\hat{0} + x_n\} = \{x_n\}$. This is to say

$$\big\{x_n\big\} \in [x] \qquad \Longleftrightarrow \qquad \big\{\,\hat{0} + x_n\big\} \in [x] \quad ,$$

or that, equivalently, there exists an additive identity element for every $x \in \mathbb{Q}$.

Example 4.2.12 With Axiom 4.2.11, we have associated every element of $C_{\mathbb{Q}}$ with the endpoint A of the real line segment **AB**. This is done because every $x \in \mathbb{Q}$ has zero fractional magnitude with respect to infinity. Therefore, we may mingle the geometric and algebraic notations to write

$$\big\{\,\hat{0} + x_n\big\} \equiv \big\{A + x_n\big\} \in [A + x] \quad .$$

By extending the line segment in consideration from $\mathbf{AB} \equiv [0, \infty]$ to $ZB \equiv [-\infty, \infty]$, the number zero is now in the center of A which is an interior point of ZB. Therefore, we may give an algebraic construction by Cauchy equivalence classes for all

$$\mathbb{R}_0^{\mathcal{X}} = \big\{\aleph_{\mathcal{X}} + b \;\big|\; b \in \mathbb{R}_0\big\} \quad ,$$

by changing the interior point attached to the sequences in the equivalence classes. For any interior point $X \in \mathbf{AB}$, there is an equivalence class $[X + x]$ such that

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \quad , \quad [x] = b \in \mathbb{R}_0 \qquad \Longrightarrow \qquad [X + x] \equiv [\aleph_{[\mathcal{X}]} + x] = \aleph_{\mathcal{X}} + b \quad .$$

In this notation, the comma is a logical "and" ($\wedge$) so the implication follows if both conditions on the left are true. Note the number $\mathcal{X}$ indicating that $\aleph_{\mathcal{X}}$ has $100 \times \mathcal{X}\%$ fractional distance with respect to **AB** is an equivalence class $\mathcal{X} = [\mathcal{X}] \subset C_{\mathbb{Q}}$ with no requisite geometric part because $0 < \mathcal{X} < 1$ implies $\mathcal{X} \in \mathbb{R}_0$.

Definition 4.2.13 In the following definitions, the sign of x is restricted appropriately for the neighborhood of the origin and the maximal neighborhood of infinity. $C_{\mathbb{Q}}^{\mathbf{AB}}$ is a Cartesian product of $C_{\mathbb{Q}}$ with the set of all $X \in \mathbf{AB}$. Specifically,

$$C_{\mathbb{Q}}^{\mathbf{AB}} = \big\{X\big\} \times C_{\mathbb{Q}} = \big\{X + [x] \;\big|\; X \in \mathbf{AB},\ [x] \subset C_{\mathbb{Q}}\big\} \quad .$$

Since it is considered desirable to give a totally algebraic construction, we may give the equivalent definition

$$C_{\mathbb{Q}}^{\mathbf{AB}} = \big\{\aleph_{\mathcal{X}}\big\} \times C_{\mathbb{Q}} = \big\{\aleph_{[\mathcal{X}]} + [x] \;\big|\; [x], [\mathcal{X}] \subset C_{\mathbb{Q}},\ 0 \leq [\mathcal{X}] \leq 1\big\} \quad .$$

In this second convention, $x \geq 0$ if $\mathcal{X} = 0$ and $x < 0$ if $\mathcal{X} = 1$. This is required for the elements of $C_{\mathbb{Q}}^{\mathbf{AB}}$ to be in $[0, \infty)$.

Remark 4.2.14 In Definition 4.2.13, the second definition $C_{\mathbb{Q}}^{\mathbf{AB}} = \big\{\aleph_{\mathcal{X}}\big\} \times C_{\mathbb{Q}}$ avoids any ambiguity related to the many-to-one relationship between points in **AB** and the numbers in the algebraic representations of those points. For instance, there is no single equivalence class of rationals containing all of $\mathbb{R}_0^0$ so there is no inherently well-defined notion of the equivalence class of a geometric point.

Definition 4.2.15 The equivalence class of a geometric point X is the equivalence class of the number in the center of its algebraic representation $\mathscr{X} \equiv X$. That is

$$\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X} \qquad \Longrightarrow \qquad [X] \equiv [\aleph_{\mathcal{X}}] = \aleph_{[\mathcal{X}]} = \aleph_{\mathcal{X}} \quad .$$

This notation is redundant because X is nothing like a Cauchy sequence. In general, we will use the $\aleph_{[\mathcal{X}]} = [\aleph_{\mathcal{X}}]$ notation. The main purpose of the present definition is to formalize the identical sameness of the two definitions of $C_{\mathbb{Q}}^{\mathbf{AB}}$ given in Definition 4.2.13.

Definition 4.2.16 Every $\aleph_{\mathcal{X}} \in \mathbb{R}_{\aleph}^{\mathcal{X}} \subset \mathbb{R}$ is a Cauchy equivalence class $\aleph_{\mathcal{X}} = [\aleph_{\mathcal{X}}] = \aleph_{[\mathcal{X}]} \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ where $\aleph_{\mathcal{X}} \in \mathbb{R}$ implies $0 \leq \mathcal{X} < 1$ so that $\mathcal{X} = [\mathcal{X}] \subset C_{\mathbb{Q}}$. If $\mathcal{D}_{\mathbf{AB}}(AX) = \mathcal{X}$, then $[X] \equiv [\aleph_{\mathcal{X}}]$.

Axiom 4.2.17 Every $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ is a Cauchy equivalence class $x = \aleph_{[\mathcal{X}]} + [b] = [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$. $\text{Big}(x)$ is defined by $[\mathcal{X}] \in C_{\mathbb{Q}}$ and $\text{Lit}(x)$ is defined by $[b] \in C_{\mathbb{Q}}$. As in Definition 4.1.5, x is defined as the sum of its big and little parts. In other words, without inventing the object $C_{\mathbb{Q}}^{\mathbf{AB}}$, we have the equivalent axiom that every $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ is an ordered pair of Cauchy equivalence classes of rationals

$$x = \big([\mathcal{X}], [b]\big) \subset C_{\mathbb{Q}} \times C_{\mathbb{Q}} \quad ,$$

where the Cartesian product is

$$C_{\mathbb{Q}} \times C_{\mathbb{Q}} : \big([\mathcal{X}], [b]\big) \to \aleph_{[\mathcal{X}]} + [b] \quad .$$

Remark 4.2.18 Axiom 4.2.17 is totally compliant with the requirement of Axiom 4.2.6 that all real numbers can be constructed from Cartesian products of subsets of $C_{\mathbb{Q}}$, or partitions thereof.

Example 4.2.19 This example gives a Cauchy equivalence class definition of $\aleph_{\mathcal{X}}$, as in Definition 4.2.16. Suppose $0 \leq x \leq 1$ and that

$$x = [x] = [\{x_n\}] = \Big\{ x_1, x_2, x_3, ... \Big\} \quad .$$

It follows that

$$\aleph_x = \aleph_{[x]} = [\aleph_x] = [\{\aleph_x^n\}] = [\aleph_{\{x_n\}}] = \Big\{ \aleph_{x_1}, \aleph_{x_2}, \aleph_{x_3}, ... \Big\} \quad ,$$

where we have moved the iterator into the superscript position at one of the intermediate steps.

Theorem 4.2.20 *If X and Y are two interior points of* $\mathbf{AB}$*, then two Cauchy equivalence classes $[X + x]$ and $[Y + y]$ are equivalent if and only if $X = Y$ and $x = y$.*

Proof. By Definition 4.2.15, we have $[X + x], [Y + y] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$. Every element of $C_{\mathbb{Q}}^{\mathbf{AB}}$ can be expressed as the Cartesian product of two elements of $C_{\mathbb{Q}}$

$$\Big\{ [\mathcal{X}] \subset C_{\mathbb{Q}} \Big\} \times \Big\{ [b] \subset C_{\mathbb{Q}} \Big\} : \big([\mathcal{X}], [b] \big) \to \aleph_{[\mathcal{X}]} + [b] \quad .$$

By the definition of the equivalence class, every element of $C_{\mathbb{Q}}$ is such that

$$[x] = [y] \qquad \Longleftrightarrow \qquad x = y \quad ,$$

so the same must be true for the ordered pairs:

$$\big([\mathcal{X}], [x] \big) = \big([\mathcal{Y}], [y] \big) \qquad \Longleftrightarrow \qquad \big(\aleph_{\mathcal{X}}, x \big) = \big(\aleph_{\mathcal{Y}}, y \big) \quad .$$

Per Definition 4.2.15, the equivalence class of $[X]$ is uniquely determined by the equivalence class of $\aleph_{\mathcal{X}}$ so it follows that $X = Y$ if and only if $[X] = [Y]$. The theorem is proven.

§4.3 The Maximal Neighborhood of Infinity

The main purpose of this section is to treat the properties of real numbers $x \in \mathbb{R}_{\aleph}^{\mathcal{X}}$ for the special case of $\mathcal{X} = 1$. Again, the reader must note that formally $\mathbb{R}_{\aleph}^1 \not\subset \{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$ due to the restriction $0 < \mathcal{X} < 1$ given by Definition 4.1.3. Whenever $\mathbb{R}_0^{\mathcal{X}}$ or $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is taken to mean $\mathcal{X} = 0$ or $\mathcal{X} = 1$, referring the neighborhood of the origin and the maximal neighborhood of infinity respectively, we will always make an explicit statement indicating $0 \not< \mathcal{X} \not< 1$.

Definition 4.3.1 The whole maximal neighborhood of infinity is

$$\mathbb{R}^1_\aleph = \left\{ \aleph_1 - b \;\middle|\; b \in \mathbb{R}^0_\aleph \right\} \quad .$$

Remark 4.3.2 We have defined $\aleph_1$ as the greatest number in the algebraic representation $\mathscr{B}$ of $B \in \mathbf{AB} \equiv [0, \infty]$. Therefore, $\aleph_1$ is an infinite element not in the real numbers. As the arithmetic of ∞ is usually defined, if we set $\aleph_1 = \infty$, then it would follow that $\infty - b = \infty$ and $\mathbb{R}^1_\aleph \cap \mathbb{R} = \varnothing$. This is not the desired behavior so we will make special notation custom tailored to deliver what is desired.

Definition 4.3.3 ∞ is called geometric infinity or simply infinity.

Definition 4.3.4 $\widehat{\infty}$ is called algebraic infinity. It shall be called infinity hat as well.

Definition 4.3.5 Additive absorption is a property of ∞ such that all $x \in \mathbb{R}$ are additive identities of ∞. The additive absorptive property is

$$\infty \pm x = \infty \quad .$$

Multiplicative absorption is a property of $\pm\infty$ such that all non-zero $x \in \mathbb{R}$ are multiplicative identities of $\pm\infty$. The multiplicative absorptive property is

$$\infty \cdot x = \begin{cases} \infty & \text{for} \quad x > 0 \\ -\infty & \text{for} \quad x < 0 \end{cases} \quad .$$

Remark 4.3.6 Note that infinity and zero are both multiplicative absorbers while zero's additive absorptive property is such that zero gets absorbed. Indeed, the contradiction inherent to mutual multiplicative absorption may be identified as a reason contributing to the canonical non-definition of the $0 \cdot \infty$ operation.

Definition 4.3.7 The symbol $\widehat{\infty}$ refers to an infinite element

$$\pm\left|\widehat{\infty}\right| = \lim_{x \to 0^\pm} \frac{1}{x} \quad , \qquad \text{and} \qquad \left|\widehat{\infty}\right| = \lim_{n \to \infty} \sum_{k=1}^{n} k \quad ,$$

together with an instruction not to perform the additive or multiplicative operations usually imbued to infinite elements.

Remark 4.3.8 What we have done in Definition 4.3.7 is exactly what we have done with $\hat{0}$ in Definition 4.2.10. In the case of $\hat{0}$, it was not in any way strange to entertain the notion that one might simply choose not to do the absorptive operations of zero and neither should the present convention for $\widehat{\infty}$ be considered in any way strange or ill-defined. In Section 4.4, we will construct an infinite element—what might be called an instance of infinity—stripped of its absorptive operations by considering the invariance of **AB** under the permutations of the labels of its endpoints. As in Sections 4.1 and 4.2, we will define some objects in the present section to facilitate a formal construction in Section 4.4.

Theorem 4.3.9 *The two open intervals* $(-\infty,\infty)$ *and* $(-\widehat{\infty},\widehat{\infty})$ *are identically equal. In other words, the real number line may be expressed identically as* $\mathbb{R}=(-\widehat{\infty},\widehat{\infty})$ *or* $\mathbb{R}=(-\infty,\infty)$.

Proof. For $a,b\in\mathbb{R}^+$, it may be taken for granted that

$$(-a,b)=(-|a|,|b|) \quad .$$

It follows, therefore, that this is true for $a,b\in\overline{\mathbb{R}}^+$. Then, per Definition 4.3.7,

$$\pm\big|\widehat{\infty}\big|=\pm\big|\infty\big| \qquad\Longrightarrow\qquad \mathbb{R}=(-\widehat{\infty},\widehat{\infty}) \quad .$$

Example 4.3.10 This example demonstrates the arithmetic constraints that would have to be placed on the limit definition of infinity if it was said to define $\widehat{\infty}$ rather than $|\widehat{\infty}|$, as in Definition 4.3.7. This example also demonstrates the general motivation for such notation by demonstrating the large burden that would imposed if the absolute value bars were absent in Definition 4.3.7. In its limit incarnation, the additive absorptive property of ∞ is demonstrated as

$$a+\infty=a+\lim_{x\to0}\frac{1}{x}=\lim_{x\to0}\frac{ax+1}{x}=\text{diverges}=\infty \quad .$$

Therefore, if the limit were said to define $\widehat{\infty}$, then the hat's arithmetic constraint "don't simplify this expression by absorption" would mean to keep a out of the limited expression. Similarly, multiplicative absorption is demonstrated as

$$a\cdot\infty=a\cdot\lim_{x\to0}\frac{1}{x}=\lim_{x\to0}\frac{a}{x}=\text{diverges}=\infty \quad .$$

In either case, the limit expression diverges in $\mathbb{R}$ and no contradiction is obtained by keeping a out of the expression to avoid it being "absorbed."

The utility in adding the hat to infinity is that it supports the notion that a number lying x units of Euclidean distance away from the least number $0=\aleph_0$ in the algebraic representation of $A\in\mathbf{AB}$ should, under permutation of the labels of the endpoints of **AB**, be mapped to another number x' lying x units of distance away from the greatest number $\aleph_1$ in the algebraic representation of $B\in\mathbf{AB}$. By suppressing the additive absorption, we let $x'=\aleph_1-x=\widehat{\infty}-x\neq\infty$. Per Definition 4.3.1, this number is $x'\in\mathbb{R}^1_\aleph$. By suppressing the multiplicative absorption of $\widehat{\infty}$, we introduce notation by which it is possible to complement Definition 4.1.1 with the statement

$$\frac{\aleph_{\mathcal{X}}}{\infty} = \mathcal{X} \qquad \Longleftrightarrow \qquad \aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad .$$

In the former part this treatise, we have demonstrated a requirement for numbers such as x' and $\aleph_{\mathcal{X}}$, and $\widehat{\infty}$ is a notation for an infinite element tailored to the requirement. Indeed, where ***algebra is called the study of mathematical symbols and the rules for manipulating them***, algebraic infinity $\widehat{\infty}$ is a perfectly ordinary algebraic object and well-defined.

Definition 4.3.11 For any $\mathcal{X} \in \mathbb{R}$, the symbol $\aleph_{\mathcal{X}}$ is defined as

$$\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty} \quad .$$

Definition 4.3.12 In terms of $\widehat{\infty}$, the whole maximal neighborhood of infinity is defined as

$$\mathbb{R}^1_{\aleph} = \left\{ \widehat{\infty} - b \;\middle|\; b \in \mathbb{R}^0_{\aleph},\; b \neq 0 \right\} \quad .$$

Definition 4.3.13 The maximal natural neighborhood of infinity is defined as

$$\mathbb{R}^1_0 = \left\{ \widehat{\infty} - b \;\middle|\; b \in \mathbb{R}^+_0 \right\} \quad .$$

§4.4 Equivalence Classes for the Maximal Natural Neighborhood of Infinity

We could easily construct $\mathbb{R}^1_0$ following the prescription in Section 4.2. There, we introduced zero as the least number in the algebraic representation of $A \in \mathbf{AB} \equiv [0, \infty]$ and then we made the extension to an arbitrary interior point by considering A as the midpoint of $ZB \equiv [-\infty, \infty]$. However, we could have left A as an endpoint and then extended the construction to the other endpoint B to define the maximal neighborhood via a symmetry argument. For breadth, here we will use a similar symmetry argument to take a slightly different approach to the Cauchy construction of the maximal neighborhood infinity. The material in the present section will constitute an independent motivation for the intermediate neighborhoods, separate from the main fractional distance approach. We will generate a non-absorbing infinite element $\widehat{\infty}$ and then we will define the $\aleph_{\mathcal{X}}$ as its fractional parts.

In Section 4.2, we defined a real number as an ordered pair of Cauchy equivalence classes of rationals: one for the big part and one for the small part. This approach required that we assume the $\aleph$ notation before we can define an equivalence class $[\aleph_{\mathcal{X}}] = \aleph_{[\mathcal{X}]} = \aleph_{\mathcal{X}}$. We were very well-motivated to assume numbers in this form, particularly by Main Theorem 3.2.6 proving that some real numbers are larger than any real number, and by Theorem 3.2.2 proving that there exists at least one real number having 50% fractional magnitude with respect to **AB**. However, it remains that $\aleph_{\mathcal{X}}$ is inherently foreign to what is called real analysis. Therefore, in the present section, we will give an alternative construction for $\mathbb{R}^1_0$ based on the geometric invariance of line segments under the permutations of the labels of their endpoints. The numbers in the maximal neighborhood of infinity are defined according to ∞: a number not at all foreign to real analysis. Then, with $|\widehat{\infty}| = \infty$ defined as in the previous section, and with a formal construction given here for the maximal neighborhood of infinity, we will use $\widehat{\infty}$ as an independent constructor for $\aleph_{\mathcal{X}}$ and the intermediate neighborhoods.

Axiom 4.4.1 A Euclidean line segment AB [1] is invariant under permutations of the labels of its endpoints, *e.g.*: $AB = BA$.

Definition 4.4.2 Define a geometric permutation operator $\hat{P}$ such that

$$\hat{P}(AB) = BA \quad .$$

Remark 4.4.3 In this section, we will construct $\mathbb{R}^1_0$ from the operation of $\hat{P}$ on Cauchy equivalence classes of rational numbers, *e.g.*: $\hat{P}([x])$. To do so, we must develop the induced operation of $\hat{P}$ on the algebraic interval representation $[a, b] \equiv AB$. (It is a pleasant coincidence that the equivalence class bracket notation is exactly consistent with the abused notion of a closed one-point interval $[x, x] = [x]$.) As in Section 4.2, our departure from the usual Cauchy construction of $\mathbb{R}$ begins with an acknowledgment that $0 \in \mathbb{Q}$ does not follow from $\mathbb{N}$. Again, we introduce $0 = \aleph_0$ as the least number in the algebraic representation of $A \in \mathbf{AB}$. Then we assume zero is an additive identity element of every $n \in \mathbb{N}$ to obtain

$$\frac{m}{n} \in \mathbb{Q} \qquad \Longrightarrow \qquad \frac{m}{n} = \frac{0+m}{n} = \frac{0}{n} + \frac{m}{n} = 0 + \frac{m}{n} \quad .$$

Finally, we will put the hat on $\hat{0}$ to remind us not to simplify the expression. The elements of $C_{\mathbb{Q}}$ now have an interpretation as Euclidean magnitudes measured relative to the origin of $\mathbb{R}$. Specifically, $\frac{m}{n}$ is an abstract element of $\mathbb{Q}$ but $\hat{0} + \frac{m}{n}$ is the rational length of a real line segment whose left endpoint has zero as the least number in its algebraic representation. This follows from Definition 4.2.15 giving $[A] = [\aleph_0] = \aleph_{[0]} = 0 = \hat{0}$.

Definition 4.4.4 The Euclidean chart x on **AB** is such that $\min(x \in A) = 0$ and $\max(x \in B) = \aleph_1$ regardless of the permutation of the labels of the endpoints. In other words, the ordering of real numbers is such that numbers nearer to B are always greater than those nearer to A.

Definition 4.4.5 Define an operator $\hat{\mathcal{P}}_0([x]; \hat{0})$ which formalizes the notion of $\hat{P}([x])$. Per Definition 4.4.2, the domain of $\hat{P}$ is not in $C_{\mathbb{Q}}$ so we introduce a special algebraic permutation operator $\hat{\mathcal{P}}_0([x];\hat{0})$ dual to $\hat{P}$ which formally operates on equivalence classes. The definition is

$$\hat{\mathcal{P}}_0 : \hat{0} \times C_\mathbb{Q} \to \widehat{\infty} \times C_\mathbb{Q} \quad ,$$

where

$$\hat{0} \times C_\mathbb{Q} = \Big\{ \hat{0} + [x] \;\Big|\; [x] \subset C_\mathbb{Q} \Big\} \quad , \quad \text{and} \qquad \widehat{\infty} \times C_\mathbb{Q} = \Big\{ \widehat{\infty} - [x] \;\Big|\; [x] \subset C_\mathbb{Q} \Big\} \quad .$$

Example 4.4.6 This example demonstrates the working of $\hat{P}$ and $\hat{\mathcal{P}}_0$ to give a formal construction of $\mathbb{R}_0^1$ by Cauchy sequences of rational numbers. Suppose $b \in \mathbb{R}_0$ is a well-defined equivalence class of rationals lying within the algebraic representation $\mathscr{A}$ of $A \in \mathbf{AB}$. Now operate on $\mathbf{AB}$ with $\hat{P}$ so that

$$\hat{P}(\mathbf{AB}) = \mathbf{BA} \quad .$$

The permutation of the labels of the endpoints has not changed the geometric position of b along the line segment. Definition 4.4.4 requires that the orientation of the Euclidean coordinate along the line segment has been reversed, so, therefore, we no longer have the property $b = [x] \subset C_\mathbb{Q}$ for the following reason. Every rational number is less than some natural number and all such numbers have zero fractional distance with respect to $\mathbf{AB}$. Before operating with $\hat{P}$, b was in the algebraic representation of the the point A but by operating with the geometric permutation operator $\hat{P}$ it becomes a number in the algebraic representation of B. The FDFs are defined such that

$$\mathcal{D}_\mathbf{AB}(\mathbf{AB}) = \mathcal{D}^\dagger_\mathbf{AB}(\mathbf{AB}) = 1 \quad ,$$

which requires that b must now have unit fractional magnitude with respect to $\mathbf{AB}$. Every $[x] \subset C_\mathbb{Q}$ has zero fractional magnitude so if $b \neq [x]$, what number is it? The number is given by

$$b = \hat{\mathcal{P}}_0(\hat{0}, [x]) = \widehat{\infty} - [x] \quad .$$

Under permutation of the labels of the endpoints of a line segment, a number having distance $[x] \subset C_\mathbb{Q}$ from one endpoint becomes another number having the same distance relative to the other endpoint.

Remark 4.4.7 We take it for granted that if there exists a real number x separated by distance L from the least number in the algebraic representation of the endpoint A of an arbitrary real line segment $AB \equiv [a, b]$—with x interior in the sense that $x \in (a, b)$—then it is guaranteed by the geometric mirror symmetry of all line segments that there must exist another real number separated from the endpoint B by the same distance L. If we bestowed $\widehat{\infty}$ with the property of additive absorption, then there would be no such number because $\hat{0} + x \to x$ but $\widehat{\infty} - x \to \widehat{\infty}$. Similarly, if there exists a real number lying one third of the way from A to B, then there must exist another real number lying one third of the way from B to A. This follows from the cut-in-a-line definition of $\mathbb{R}$ given by Definition 2.1.5. For the case of **AB**, it will be impossible to express these third fraction numbers if $\widehat{\infty}$ has the property of multiplicative absorption. Since the third numbers *must* exist, $\aleph_{\mathcal{X}}$ *does* exist. Therefore, the existence of an instance of infinity devoid of any absorptive properties is absolutely granted if the mirror symmetry of a geometric line segment is to be preserved in its interpretation as an algebraic interval of numbers.

Our thesis is that we should preserve the underlying geometric construction of $\mathbb{R}$ without invoking a contradictory algebraic construction. Under this thesis, $\widehat{\infty}$ is forced into existence. Often times, the position is taken that infinity is absolutely absorptive due to the limit definition of infinity and the attendant absorptive properties of limits (Example 4.3.10.) As an indirect consequence of such reasoning, the mirror symmetry of line segments must be rejected in the algebraic realm of mathematics. But why should it be preferred that the algebraic construction overrides the geometric construction? Is it not equally valid to override the algebraic construction with the geometric one? Considering the history of mathematics, it is, in the opinion of this writer, far more appropriate to preserve the geometric construction at all costs. It is very easy to do so when the symbol $\widehat{\infty}$ is given by the limit definition of infinity as

$$\lim_{x\to 0^{\pm}} \frac{1}{x} = \pm\big|\widehat{\infty}\big| \quad ,$$

without $\widehat{\infty}$ itself being interchangeably equal with the limit expression. Furthermore, this scheme is such that the algebraic and geometric concepts are complementary without requiring that one override the other.

In Definition 4.2.17, we gave the definition of $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ in terms of ordered pairs of elements of $C_{\mathbb{Q}}$. The purpose of the present alternative treatment for the maximal neighborhood $\mathbb{R}_0^1$ is not to replace that definition but to complement it with a different equivalence class construction for the maximal neighborhood from which the constructions of the intermediate neighborhoods may be extracted. In this present section, we have used the permutation operator $\hat{P}$ which is quite similar to the implicit translation operator by which we were able to attach elements of $C_{\mathbb{Q}}$ to different interior points of **AB** in Section 4.2. The main utility in developing the idea of a number in the neighborhood of infinity as the operation of $\hat{\mathcal{P}}_0$ on an equivalence class of rationals is that it independently generates the requirement for an infinite element lacking the usual absorptive properties of infinity. With $\widehat{\infty}$ granted, it gives a separate means by which we may construct the $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ without invoking the direct ordered pair definition: the $\aleph_{\mathcal{X}}$ in such numbers are the fractions of the non-absorbing infinite element $\widehat{\infty}$.

Definition 4.4.8 Every $x \in \mathbb{R}_0^1$ is defined as the output of $\hat{\mathcal{P}}_0$ operating on an element of $C_{\mathbb{Q}}$. This is the Cauchy equivalence class construction of real numbers in the maximal natural neighborhood of infinity.

Example 4.4.9 In this example, we complement the separate definitions for ∞ and $\widehat{\infty}$ heretofore given. We will show, for example, how they might be more fully conceptually distinguished as two mutually distinct kinds of infinite elements with markedly different qualia beyond their separate technical definitions. While we will offer these qualia as an example, we will not alter the technical definitions with the supplemental considerations proposed here. To that end, it is sometimes claimed, without proof, that one cannot place endpoints at the ends of $\mathbb{R} = (-\infty, \infty)$ because the notion of an endpoint contradicts the notion of the infinite geometric extent of a line extending infinitely far in both directions. Infinite geometric extent is the main principle that we will look at in this example.

Suppose geometric infinity ∞ is a number which cannot be included as an endpoint without contradicting the notion of the infinite geometric extent of a number line. Definition 2.1.2 defines a number line as a 1D metric space in the Euclidean metric

$$d(x, y) = \big|y - x\big| \quad .$$

If we included geometric infinity as an endpoint, then we could invoke the invariance of line segments under permutations of their endpoints to demonstrate a contradiction. Given

$$(x, y) = (x_0, y_0) \quad , \quad \text{and} \qquad (\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = (\infty - x_0, \infty - y_0) \quad ,$$

not only do the points lose their unique identity when attached to B instead of A, but if we put $(\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0))$ into the Euclidean metric, then we get

$$d(\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = \Big|\infty - x_0 - \big(\infty - y_0\big)\Big| = \big|\infty - \infty\big| = \text{undefined} \quad .$$

Clearly, this does not gel well with our intention to define a number line as a line equipped with a metric. The line is supposed to have some metrical distance between any two points but now, under the permutation of the labels A and B, we find two points that don't even have vanishing distance between them. The distance has become undefined even though this does not follow from the invariance of Euclidean line segments under such permutations.

Algebraic infinity is a number which avoids all of the problems here listed. Under permutation, we have

$$(x, y) = (x_0, y_0) \quad , \quad \text{and} \qquad (\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = (\widehat{\infty} - x_0, \widehat{\infty} - y_0) \quad .$$

Jumping ahead to the arithmetic of such numbers axiomatized in Section 5.2, we find

$$d(\hat{\mathcal{P}}_0(x_0), \hat{\mathcal{P}}_0(y_0)) = \Big|\widehat{\infty} - x_0 - \big(\widehat{\infty} - y_0\big)\Big| = \big|y_0 - x_0\big| = d(x_0, y_0) \quad ,$$

exactly as expected. The only issue which remains is to revisit is the construction for $\mathbf{AB} \equiv [0, \infty]$ that we have given by a conformal chart $x = \tan(x')$ on the line segment $AB \equiv [0, \frac{\pi}{2}]$ whose endpoints unquestioningly exist in any frame of standard analysis. For this, we propose a semantic convention to distinguish the geometric infinite element ∞ from the algebraic one $\widehat{\infty}$. Let algebraic infinity be such that it can be embedded in a larger space but let geometric infinity be such that it is totally maximal and cannot be embedded in something larger than itself. For example, the interval $[0, \frac{\pi}{2}] \subset [-\pi, \pi]$ is such that the conformal chart which sends $\frac{\pi}{2}$ to an infinite element implicitly places that element within the parent interval $[-\pi, \pi]$. The convention proposed here would require that the infinite element to which $\frac{\pi}{2}$ is conformally mapped must be algebraic infinity $\widehat{\infty}$. If we take the convention that geometric infinity ∞ is always totally geometrically maximal, then that would forbid its existence on the interior of the interval $[-\pi, \pi]$ which contains points to the right of $\frac{\pi}{2}$. In a formal adoption of the distinctions made here, one would examine the merits of a supplemental transfinite ordering relation $\widehat{\infty} < \infty$.

Remark 4.4.10 If we wish to construct $\mathbf{AB} \equiv [0, \widehat{\infty}]$ directly from $AB \equiv [0, \frac{\pi}{2}]$ as in Example 2.3.22, wherein we cite the limit definition of infinity (Definition 2.2.2) as motivating the identity

$$\tan\left(\frac{\pi}{2}\right) = \infty \quad ,$$

then we need to make rigorous the relationship between ∞ and $\widehat{\infty}$. This was the purpose of Theorem 4.3.9 proving $\mathbb{R} = (-\widehat{\infty}, \widehat{\infty})$. Since the absolute value, or the magnitude, of $\widehat{\infty}$ is the same as that of ∞, the algebraic intervals $[a, \infty]$ and $[a, \widehat{\infty}]$ must be the same interval. Though we cannot directly construct $[0, \widehat{\infty}]$ from $[0, \frac{\pi}{2}]$, we may indirectly construct it by using the limit definition of infinity to write

$$\lim_{\theta \to \frac{\pi}{2}} \tan(\theta) = \lim_{\theta \to \frac{\pi}{2}} \frac{\sin(\theta)}{\cos(\theta)} = \lim_{\substack{x \to 0 \\ y \to 1}} \frac{y}{x} = \infty \quad .$$

Now we may directly infer the existence of conformal $\mathbf{AB} \equiv [0, \widehat{\infty}]$ from the assumed interval $[0, \frac{\pi}{2}]$. Due to the transitivity of the equivalence relation, however, we must be very careful about the definition of $\widehat{\infty}$. Note that Definition 4.3.7 gives

$$\left|\widehat{\infty}\right| = \infty = \lim_{x \to 0} \frac{1}{x} \qquad \not\Rightarrow \qquad \widehat{\infty} = \lim_{x \to 0} \frac{1}{x} \quad .$$

Therefore, we must be careful about whether $\aleph_1$ is equal to geometric infinity or algebraic. If we take the convention that geometric infinity ∞ is imbued with the notion of infinite geometric extent such that an infinite line cannot have an endpoint there, as in Example 4.4.9, then we should not let $\aleph_1$ be defined by ∞ when it is said to be the greatest number in the algebraic representation of the endpoint $B \in \mathbf{AB}$. Due to the possibility of constructing $\mathbf{AB}$ from any other line segment by one conformal chart transformation or another, $\mathbf{AB}$

ought to be taken as $[0, \widehat{\infty}] = [0, \aleph_1]$ in the absence of explicit words to the contrary.

Definition 4.4.11 The symbol $\aleph_1$ is an alternative notation for algebraic infinity. We have

$$\aleph_1 = \widehat{\infty} \quad , \quad \text{and} \qquad \aleph_1 \neq \infty \quad .$$

Remark 4.4.12 All of the contradictions which forbid additive and multiplicative inverses for ∞ stem from its limit definition. Should we bestow, then, these inverses on $\widehat{\infty} = \aleph_1$? To the extent that the notion of fractional distance requires that $100\% - 100\% = 0\%$, or that $100\%/100\% = 1$, the answer is yes. Similarly, all of the contradictions which disallow a definition for the operation $0 \cdot \infty$ are rooted in the limit definition of infinity. Note that $0 \cdot \widehat{\infty} = \aleph_0 = 0$ follows as a special of $\aleph_\mathcal{X} = \mathcal{X} \cdot \widehat{\infty}$. We should not expect any contradictions because $\widehat{\infty} \neq \infty$ and the limit definition is out of scope.

Axiom 4.4.13 $\aleph_1$ is such that

$$\aleph_1 - \aleph_1 = 0 \quad , \quad \text{and} \qquad \frac{\aleph_1}{\aleph_1} = 1 \quad .$$

Theorem 4.4.14 *The maximal whole neighborhood of infinity is a subset of the real numbers.*

Proof. Taking for granted that $x \in \mathbb{R}^1_\aleph$ does not have any infinitesimal part, which is obvious, it suffices to show the compliance with Definition 2.1.5: a real number $x \in \mathbb{R}$ is a cut in the real number line. Compliance follows directly from Definition 4.3.12 giving

$$\mathbb{R}^1_\aleph = \left\{ \widehat{\infty} - b \;\middle|\; b \in \mathbb{R}^0_\aleph,\ b \neq 0 \right\} \quad .$$

We clearly have

$$(-\infty, \infty) = (-\infty, \aleph_\mathcal{X} + b] \cup (\aleph_\mathcal{X} + b, \infty) \quad .$$

Even though we do not yet have an equivalence class construction of $b \in \mathbb{R}^0_\aleph \setminus \mathbb{R}_0$, it is obvious that $\widehat{\infty} - b$ is a cut in the real number line because b, whatever its algebraic construction, is such that it has less than unit zero fractional magnitude with respect to **AB** and is also such that $b > 0$. (The intuitive ordering assumed in this theorem is formalized in Axiom 5.2.14.) ∅

Corollary 4.4.15 *All numbers $x \in \{\mathbb{R}^\mathcal{X}_\aleph\}$ are real numbers.*

Proof. The ordering of $\mathbb{R}$ given by Axiom 3.1.13 is such that $0 < \mathcal{X} < 1$ guarantees

$$(0, \infty) = (0, \aleph_\mathcal{X} \pm b] \cup (\aleph_\mathcal{X} \pm b, \infty) \quad .$$

Definition 2.1.5 is satisfied trivially and the theorem is proven. ✐

Remark 4.4.16 As a final aside in this section, note the curious condition under which algebraic infinity $\aleph_1$ has its foundation in the geometric properties of a line segment while geometric infinity ∞ has its foundation in the limit of an algebraic expression. The reciprocity among these two constructions of an infinite element might indicate some deeply fundamental issues extending beyond the semantic convention of our having chosen to call one infinite element geometric and the other algebraic. We will not proceed along that analytical direction but the reciprocity of the cross-sampling of the concepts is interesting and tantalizing.

§5 Arithmetic

§5.1 Operations for Infinite Elements

Here we give arithmetic operations for $\infty, \widehat{\infty} \notin \mathbb{R}$ to support the axioms for real numbers $x \in \mathbb{R}$ with non-zero big parts to appear in Section 5.2.

Axiom 5.1.1 The operations for $\infty \neq \aleph_1$ with $b \in \mathbb{R}_0^+$ are

$$\begin{aligned} \infty \pm b &= \infty \\ \infty \pm \big(-b \big) &= \infty \\ -\big(\pm \infty \big) &= \mp \infty \\ \infty \cdot b &= \infty \\ \frac{\infty}{b} &= \infty \\ \frac{b}{\infty} &= 0 \quad . \end{aligned}$$

Axiom 5.1.2 We give the following supplemental axioms for zero and ∞.

$$\begin{aligned} \infty + 0 = 0 + \infty &= \infty \\ \infty \cdot 0 = 0 \cdot \infty &= \text{undefined} \\ \frac{\infty}{0} &= \text{undefined} \\ \frac{0}{\infty} &= 0 \quad . \end{aligned}$$

Axiom 5.1.3 The operations for $\widehat{\infty} = \aleph_1$ with $b \in \mathbb{R}_0^+$ are

$$\begin{aligned} \widehat{\infty} \pm b &= \pm b + \widehat{\infty} \\ \widehat{\infty} \pm \big(-b \big) &= \widehat{\infty} \mp b \\ -\big(\pm \widehat{\infty} \big) &= \mp \widehat{\infty} \end{aligned}$$

$$\widehat{\infty} \cdot b = b \cdot \widehat{\infty} = \aleph_b$$
$$\frac{\widehat{\infty}}{b} = \aleph_{(b^{-1})}$$
$$\frac{b}{\widehat{\infty}} = 0 \quad .$$

Axiom 5.1.4 We give the following supplemental axioms for zero and $\widehat{\infty}$.

$$\widehat{\infty} + 0 = 0 + \widehat{\infty} = \widehat{\infty}$$
$$\widehat{\infty} \cdot 0 = 0 \cdot \widehat{\infty} = 0$$
$$\frac{0}{\widehat{\infty}} = 0$$
$$\frac{\widehat{\infty}}{0} = \text{undefined} \quad .$$

Remark 5.1.5 The most important facet of Axiom 5.1.4 is the $0\cdot\widehat{\infty}$ operation contrary to the undefined $0 \cdot \infty$ operation (Axiom 5.1.2.) This is required to preserve the notion of fractional distance: zero times 100% is 0%. To facilitate this definition, it will be required that we define division as a separate operation distinct from multiplication by an inverse. This will be one of the major distinctions of the axioms of Section 5.2 from the well-known field axioms. We demonstrate the principle in Example 5.1.6.

Example 5.1.6 This example gives a common argument in favor of the non-definition of a product between an infinite element and zero. Then we will show how the contradiction is avoided by taking away the assumed associativity among multiplication and division. Suppose $c \in \mathbb{R}^0_\aleph$ so that

$$\frac{c}{\widehat{\infty}} = 0 \quad .$$

Now suppose $0 \cdot \widehat{\infty}$ is a defined operation so that

$$z = 0 \cdot \widehat{\infty} \quad .$$

Substitute $\frac{c}{\widehat{\infty}} = 0$ and use the $\frac{\widehat{\infty}}{\widehat{\infty}} = 1$ property of Axiom 4.4.13 to obtain by association of multiplication and division the expression

$$z = 0 \cdot \widehat{\infty} = \frac{c}{\widehat{\infty}} \cdot \widehat{\infty} = c \cdot \frac{\widehat{\infty}}{\widehat{\infty}} = c \quad .$$

This shows that $0 \cdot \widehat{\infty}$ is not a well-defined operation because $z = c$ is not a unique output. When we define division as a third operation beyond multiplication and addition, we should not assume associativity among the distinct divisive and multiplicative operations, and neither will we axiomatize it in Section 5.2. Without assumed associativity among the terms, we cannot show that z fails to be a well-defined output of the product $0 \cdot \widehat{\infty}$. In that case, we will assume there is no problem with the definition $0 \cdot \widehat{\infty} = 0$.

Axiom 5.1.7 For any $x \in \mathbb{R}$, we have

$$x^{\widehat{\infty}} = \begin{cases} \infty & \text{for} \quad x > 1 \\ 1 & \text{for} \quad x = 1 \\ 0 & \text{for} \quad 0 \leq x < 1 \end{cases} \quad .$$

The product of an infinite number of finite numbers greater than one is absolutely absorptive.

Remark 5.1.8 In the remainder of this section, we will motivate Axiom 5.1.7 giving $x^{\widehat{\infty}} = \infty$ as opposed to the alternative convention $x^{\widehat{\infty}} = \widehat{\infty}$. In general, for $x \neq y$, we would not expect that $x^{\aleph_1} = y^{\aleph_1} = \aleph_1 = \widehat{\infty}$. Since $\infty \neq \aleph_1$, by choosing the convention of Axiom 5.1.7 we sidestep the notion that two different numbers x and y raised to the same power might both land precisely at $\aleph_1$.

Theorem 5.1.9 *For $k \neq 1$, we have $\widehat{\infty}^k \neq \widehat{\infty}$.*

Proof. To prove this theorem, it will suffice to prove that

$$\widehat{\infty}^2 = \widehat{\infty} \cdot \widehat{\infty} = \aleph_1 \cdot \aleph_1 \neq \aleph_1 \quad .$$

Definition 4.3.11 requires that for any $x \in \mathbb{R}$, the symbol $\aleph_x$ is defined as

$$\aleph_x = x \cdot \widehat{\infty} \quad .$$

Choose $x = \aleph_{\mathcal{X}}$ such that $0 \leq \mathcal{X} < 1$. Then

$$\mathcal{X} \cdot \widehat{\infty}^2 = \mathcal{X} \cdot \widehat{\infty} \cdot \widehat{\infty} = \aleph_{\mathcal{X}} \cdot \widehat{\infty} = \aleph_{\aleph_{\mathcal{X}}} \quad .$$

If $\widehat{\infty}^2 = \widehat{\infty}$, however, then we could write

$$\mathcal{X} \cdot \widehat{\infty}^2 = \mathcal{X} \cdot \widehat{\infty} = \aleph_{\mathcal{X}} \quad .$$

Since $\aleph_{\mathcal{X}} \in \mathbb{R}$, it cannot be equal to the number $\aleph_{\aleph_{\mathcal{X}}} \notin \mathbb{R}$ which has much greater than unit fractional distance with respect to infinity. This proves the theorem.

Theorem 5.1.10 *The operation $x^{\widehat{\infty}} = \widehat{\infty}$ is not well-defined.*

Proof. Assume $0 < \mathcal{X} < 1$ and consider two expressions

$$x^{\aleph_{\mathcal{X}}+b} = \left(x^{\mathcal{X}}\right)^{\widehat{\infty}} x^b = \widehat{\infty} x^b \quad , \quad \text{and} \qquad x^{\aleph_{\mathcal{X}}+b} = \left(x^{\widehat{\infty}}\right)^{\mathcal{X}} x^b = \widehat{\infty}^{\mathcal{X}} x^b \quad .$$

By Theorem 5.1.9, we have $x^b \widehat{\infty} \neq x^b \widehat{\infty}^{\mathcal{X}}$. This proves the theorem.

Remark 5.1.11 Note the contradiction derived in Theorem 5.1.10 is avoided in the convention of Axiom 5.1.7. We have

$$x^{\aleph_\mathcal{X}+b}=\left(x^\mathcal{X}\right)^{\widehat{\infty}}x^b=\infty x^b=\infty \quad , \quad \text{and} \qquad x^{\aleph_\mathcal{X}+b}=\left(x^{\widehat{\infty}}\right)^\mathcal{X}x^b=\infty^\mathcal{X}x^b=\infty \quad .$$

Here we have relied on the usual understanding that the multiplicative absorptive property of ∞ is such all powers of ∞ are identically equal to ∞. This exceeds the definition of absorption given in Definition 4.3.5 such that ∞ absorbs $x\in\mathbb{R}$ but it is standard to set all powers of ∞ equal to ∞.

§5.2 Arithmetic Axioms for Real Numbers in Natural Neighborhoods

When one defines $\mathbb{R}$ such that the set $\mathcal{R}=\{\mathbb{R},+,\times,\leq\}$ conforms the field axioms, it is a natural progression to prove that Cauchy equivalence classes satisfy the field axioms. We do *not* presently presume that $\mathbb{R}$ is such that $\mathcal{R}$ obeys the field axioms so we will not make any such proofs. Instead, we will list the axiomatized arithmetic operations obeyed by real numbers whose little parts are less than some natural number. For disambiguation with the well-known "field axioms," the axioms given in this section are called the "arithmetic axioms." In Section 5.3, we will make proofs of certain operations given in these arithmetic axioms, and give examples. In Section 5.5, we will define the operations in terms of the numbers' underlying equivalence classes. All of the axioms given here pertain only to the natural neighborhoods $\mathbb{R}_0^\mathcal{X}$. When we give the treatment leading to $\mathbb{R}_\aleph^\mathcal{X}\backslash\mathbb{R}_0^\mathcal{X}=\varnothing$ (Section 7.4), these axioms will be fairly comprehensive. However, when we impose connectedness on $\mathbb{R}$ in Section 7.5, we will find that these axioms are not *totally* comprehensive.

The equivalence class constructions given in Section 4 were only for natural neighborhoods and here we will follow with the axiomatized arithmetic for the elements of those neighborhoods. Almost everything about the field axioms shall be preserved in the natural neighborhoods. The major exception is that we will not enforce the global closure of $\mathbb{R}$ under its operations. Among the other departures from the field axioms will be the identification of division as an operation separate from its usual definition in terms of multiplication by an inverse. Closure is nice for group theoretical applications but it is not needed for most applications in arithmetic. For example, the set $\{3,4,5\}$ is not closed under addition and yet it remains a perfectly sound algebraic structure with which one may do summation mathematics in the usual way. If one were to claim, "Non-closure doesn't break arithmetic because $\{3,4,5;+\}$ is a subset of $\{\mathbb{R};+\}$ which is an algebraic group as defined by the field axioms," then we could make an easy rebuttal by defining a set $\mathbb{T}$ to be

$$\mathbb{R}\subset\mathbb{T}=\Big\{x\;\Big|\;-\aleph_\infty<x<\aleph_\infty\Big\} \quad .$$

Then the present convention for non-closed $\{\mathbb{R};+\}$ defined with the Euclidean magnitude and supplemental arithmetic axioms is such that $\{\mathbb{R};+\}$ is a subset of the closed additive group of 1D transfinitely continued real numbers $\{\mathbb{T};+\}$.

Axiom 5.2.1 All $\mathbb{R}_0$ numbers obey the well-known axioms of a complete ordered field: Axioms 5.4.2, 5.4.4, and 5.4.7.

Remark 5.2.2 To make a distinction between the intermediate neighborhoods of infinity and the maximal neighborhood, in this section we will use the symbol $\widehat{\infty}$ rather than the symbol $\aleph_1$. However, the reader should note that the arithmetic of the maximal neighborhood follows from the arithmetic of the intermediate neighborhoods as a special case of $\aleph_{\mathcal{X}}$ with $\mathcal{X} = 1$.

Axiom 5.2.3 Addition is commutative and associative. There exists an additive identity element 0 and an additive inverse x^{-1} for every $x \in \mathbb{R}$. The operations for + are given as follows when $a, b, x, y \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$.

$+$	0	$y \in \mathbb{R}_0$	$\big(\aleph_{\mathcal{Y}} + a\big) \in \mathbb{R}_0^{\mathcal{X}}$	$\big(\widehat{\infty} - \lvert a \rvert\big) \in \mathbb{R}_0^1 \cup \widehat{\infty}$
0	0	y	$\aleph_{\mathcal{Y}} + a$	$\widehat{\infty} - \lvert a \rvert$
x	x	$x + y$	$\aleph_{\mathcal{Y}} + \big(a + x\big)$	$\widehat{\infty} - \big(\lvert a \rvert - x\big)$
$\big(\aleph_{\mathcal{X}} + b\big)$	$\aleph_{\mathcal{X}} + b$	$\aleph_{\mathcal{X}} + \big(b + y\big)$	$\aleph_{(\mathcal{X}+\mathcal{Y})} + \big(b + a\big)$	$\aleph_{(\mathcal{X}+1)} + \big(b - \lvert a \rvert\big)$
$\big(\widehat{\infty} - \lvert b \rvert\big)$	$\widehat{\infty} - \lvert b \rvert$	$\widehat{\infty} - \big(\lvert b \rvert - y\big)$	$\aleph_{(1+\mathcal{Y})} - \big(\lvert b \rvert - a\big)$	$\aleph_2 - \big(\lvert b \rvert + \lvert a \rvert\big)$

Remark 5.2.4 The most important property given by Axiom 5.2.3 is

$$\aleph_{\mathcal{X}} + \aleph_{\mathcal{Y}} = \aleph_{(\mathcal{X}+\mathcal{Y})} \quad .$$

This equality follows from the geometric notion of addition. If, for instance, $\aleph_{\mathcal{X}}$ is a number with 10% fractional distance along **AB** and $\aleph_{\mathcal{Y}}$ is a number with 20% fractional distance, then it follows that their sum is a number with 30% fractional distance along **AB**. Axiom 5.2.3 makes clear that $\mathbb{R}$ does not satisfy the usual understanding that the reals are closed under their operations. Any number $\aleph_{\mathcal{X}} + b$ with $\mathcal{X} > 1$ is not a real number, *e.g.*: the sum of two positive numbers with 99% fractional magnitude is not a real number. No x with big part $\aleph_{1.98}$ can be $x \in \mathbb{R}$.

Axiom 5.2.5 Multiplication is commutative and associative, and it is distributive over addition. It is not associative with division (which shall not be defined as multiplication by an inverse.) There exists a multiplicative identity $1 \neq 0$ for every $x \in \mathbb{R}$ but there does not exist a multiplicative inverse for all $x \in \mathbb{R}$. The operations for $\{\cdot\} = \{\times\}$ are given as follows when $a, b \in \mathbb{R}_0$, $x, y \in \mathbb{R}_0^+$, and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$.

$\times$	0	∓ 1	$y \in \mathbb{R}_0^+$	$\big(\aleph_\mathcal{Y} + a\big) \in \mathbb{R}_0^\mathcal{X}$	$\big(\widehat{\infty} - \lvert a \rvert\big) \in \mathbb{R}_0^1 \cup \widehat{\infty}$
0	0	0	0	0	0
± 1	0	-1	$\pm y$	$\aleph_{(\pm\mathcal{Y})} \pm a$	$\pm\widehat{\infty} \mp \lvert a \rvert$
x	0	$\mp x$	xy	$\aleph_{(x\mathcal{Y})} + ax$	$\aleph_x - \lvert a \rvert x$
$\big(\aleph_\mathcal{X} + b\big)$	0	$\aleph_{(\mp\mathcal{X})} \mp b$	$\aleph_{(\mathcal{X}y)} + by$	$\aleph_{(\aleph_{\mathcal{X}\mathcal{Y}} + a\mathcal{X} + b\mathcal{Y})} + ba$	$\aleph_{(\aleph_\mathcal{X} - \lvert a \rvert \mathcal{X} + b)} - b\lvert a \rvert$
$\big(\widehat{\infty} - \lvert b \rvert\big)$	0	$\mp\widehat{\infty} \pm \lvert b \rvert$	$\aleph_y - \lvert b \rvert y$	$\aleph_{(\aleph_\mathcal{Y} + a - \lvert b \rvert \mathcal{Y})} - \lvert b \rvert a$	$\aleph_{(\widehat{\infty} - \lvert a \rvert - \lvert b \rvert)} + \lvert ba \rvert$

Remark 5.2.6 The most important property given in Axiom 5.2.5 is

$$\pm\aleph_\mathcal{X} = \aleph_{(\pm\mathcal{X})} \quad .$$

This operation follows from

$$\aleph_\mathcal{X} = \mathcal{X} \cdot \widehat{\infty} \qquad \Longrightarrow \qquad \pm\aleph_\mathcal{X} = \pm\big(\mathcal{X} \cdot \widehat{\infty}\big) = \big(\pm\mathcal{X}\big) \cdot \widehat{\infty} = \aleph_{(\pm\mathcal{X})} \quad .$$

This shows that multiplication is axiomatically associative.

Remark 5.2.7 Certain of the products in Axiom 5.2.5 rely on Axiom 5.2.3. For instance, the value in the lower right corner of the multiplication table is computed as

$$\begin{aligned}\big(\widehat{\infty} - |b|\big)\big(\widehat{\infty} - |a|\big) &= \widehat{\infty} \cdot \widehat{\infty} - |b|\widehat{\infty} - |a|\widehat{\infty} + |ba| \\ &= \aleph_1 \cdot \aleph_1 - |a|\aleph_1 - |b|\aleph_1 + |ba| \\ &= \aleph_{(\aleph_1)} - \aleph_{|a|} - \aleph_{|b|} + |ba| \\ &= \aleph_{(\aleph_1)} - \big(\aleph_{|a|} + \aleph_{|b|}\big) + |ba| \\ &= \aleph_{\widehat{\infty}} + \aleph_{(-|a|-|b|)} + |ba| \\ &= \aleph_{(\widehat{\infty}-|a|-|b|)} + |ba| \quad .\end{aligned}$$

Furthermore, it follows from Axioms 5.2.3 and 5.2.5 that

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad .$$

This is the primary operation behind the original ideation for a non-absorptive infinite element. If a and b are two numbers at distances a and b respectively from the endpoint 0 of the interval $[0, \infty]$, then their difference $a - b$ must be equal (up to a sign) to the difference of two numbers lying at distances a and b from the endpoint ∞ of the same interval.

Example 5.2.8 The purpose of this example is to demonstrate that even while numbers greater than $\widehat{\infty}$ do not exist in real analysis, expressions implying the existence of such are numbers are generally not considered contradictory. Consider the quadratic equation

$$ax^2 + bx + c = 0 \quad ,$$

having roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

For every case in which $4ac > b^2$, the number x does not exist in real analysis and yet it is never claimed that the quadratic formula is contradictory. Instead, we claim that there must exist an imaginary number $i \notin \mathbb{R}$ with the property $i = \sqrt{-1}$. Therefore, the principle of fractional distance should support a conclusion that there exist transfinite numbers $x \notin \mathbb{R}$ with the property that $x > \widehat{\infty}$. We have seen the existence of such numbers implied previously when examining algebraic infinity as the endpoint of a line segment embedded in a line extending infinitely far in both directions. If we use $x = \tan(x')$ to define $\mathbf{AB} \equiv [0, \widehat{\infty}]$ on $AB \equiv [0, \frac{\pi}{2}]$, and if a number is a cut in a line as per Definition 2.1.5, then there should exist non-real transfinite numbers which are cuts in an infinite line to the right of $x = \widehat{\infty}$ in the algebraic representation of the point B.

Remark 5.2.9 When the field axioms give the arithmetic operations of $\mathbb{R}$, the difference operations follow from the sum operations as the addition of a product with -1. The $\div$ operations usually follow from the $\times$ operations as multiplication by an inverse. Presently we may define the difference operations accordingly but we may not do so for the quotient operations. As demonstrated in Example 5.1.6, the preservation of the respective geometric notions of the algebraic operations requires that $\{+, \times, \div\}$ is a set of three distinct arithmetic operations among which there is not mutual associativity. Obviously, this is a major distinction of the present axioms from the field axioms. However, Axiom 5.2.1 grants that $x \in \mathbb{R}_0$ obey the usual field axioms so there is an implicit axiom regarding a limited associativity of $\{\times, \div\}$ which we will make explicit in Axiom 5.2.10.

Axiom 5.2.10 Division and multiplication are mutually associative for any $x \in \mathbb{R}_0$. That is, all factors which are elements of $\mathbb{R}_0$ may be moved into or out of quotients and products in the usual way, even if those quotients and products contain $x \notin \mathbb{R}_0$.

Axiom 5.2.11 The operations for $\div$ are given as follows when $a, b \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$. There exists a divisive identity $1 \neq 0$ for every $x \in \mathbb{R}$. It is the same as the multiplicative identity. There exists at least one divisive inverse for every non-zero $x \in \mathbb{R}$. In this table, the row value is the numerator and the column value is the denominator.

$\div$	0	$y \in \mathbb{R}_0$	$\big(\aleph_\mathcal{Y} + a\big) \in \mathbb{R}_0^\mathcal{X}$	$\big(\widehat{\infty} - \lvert a\rvert\big) \in \mathbb{R}_0^1 \cup \widehat{\infty}$
0	nan	0	0	0
x	nan	$\frac{x}{y}$	0	0
$\big(\aleph_\mathcal{X} + b\big)$	nan	$\aleph_{(\mathcal{X}y^{-1})} + \frac{b}{y}$	$\frac{\mathcal{X}}{\mathcal{Y}}$	$\mathcal{X}$
$\big(\widehat{\infty} - \lvert b\rvert\big)$	nan	$\aleph_{(y^{-1})} - \frac{\lvert b\rvert}{y}$	$\frac{1}{\mathcal{Y}}$	1

Example 5.2.12 This example demonstrates that the quotient operations given by Axiom 5.2.11 are well-defined. (This is proven rigorously in Main Theorem 5.5.12.) An operation is well-defined if it generates a unique output. It is obvious in Axiom 5.2.11 that each operation has one and only one output. It is foreign to the usual understanding of the arithmetic of real numbers, however, that the operands giving the unique resultants are not themselves unique. Consider

$$\frac{\aleph_\mathcal{X} + b}{\aleph_\mathcal{Y} + a} = \frac{\mathcal{X}}{\mathcal{Y}} \quad .$$

If multiplication was associative with division, and vice versa, then we could multiply both sides by $\aleph_\mathcal{Y} + a$ to obtain a contradiction of the form

$$\frac{\aleph_\mathcal{X} + b}{\aleph_\mathcal{Y} + a} \cdot \big(\aleph_\mathcal{Y} + a\big) = \frac{\mathcal{X}}{\mathcal{Y}} \cdot \big(\aleph_\mathcal{Y} + a\big)$$
$$\aleph_\mathcal{X} + b = \aleph_\mathcal{X} + \frac{\mathcal{X}a}{\mathcal{Y}} \quad .$$

This is false whenever $b \neq \frac{\mathcal{X}a}{\mathcal{Y}}$ but it is not possible to show this contradiction without assuming associativity among $\{\times, \div\}$.

Example 5.2.13 This example demonstrates another immediate contradiction should we assume associativity among multiplication and division. Axiom 5.2.11 gives

$$\frac{\aleph_\mathcal{Y}}{\aleph_\mathcal{X}} = \frac{\mathcal{Y}}{\mathcal{X}} \quad , \quad \text{and} \qquad \frac{1}{\aleph_\mathcal{X}} = 0 \quad .$$

If we bestow the associativity, then

$$\frac{\aleph_\mathcal{Y}}{\aleph_\mathcal{X}} = \aleph_\mathcal{Y} \cdot \frac{1}{\aleph_\mathcal{X}} = \aleph_\mathcal{Y} \cdot 0 = 0 \neq \frac{\mathcal{Y}}{\mathcal{X}} \quad .$$

Axiom 5.2.14 The ordering of $\mathbb{R}$ is given as follows when $a, b, c, d, x, y \in \mathbb{R}_0$ and $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$. For the table, it is granted that

$$a > b \quad , \quad c > d > 0 \quad , \quad x > y \quad , \quad \text{and} \qquad \mathcal{X} > \mathcal{Y} \quad .$$

This table is such that the row identity is on the left of the given relation and the column identity is on the right.

$\leq$	$y \in \mathbb{R}_0$	$\big(\aleph_{\mathcal{Y}} + b\big) \in \mathbb{R}_0^{\mathcal{Y}}$	$\big(\aleph_{\mathcal{X}} + b\big) \in \mathbb{R}_0^{\mathcal{X}}$	$\big(\widehat{\infty} - \lvert d \rvert\big) \in \mathbb{R}_0^1$	$\widehat{\infty}$
x	$>$	$<$	$<$	$<$	$<$
$\big(\aleph_{\mathcal{X}} + a\big)$	$>$	$>$	$>$	$<$	$<$
$\big(\widehat{\infty} - \lvert c \rvert\big)$	$>$	$>$	$>$	$<$	$<$

Theorem 5.2.15 *Real numbers in the intermediate natural neighborhoods of infinity $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ do not have a multiplicative inverse.*

Proof. A number x^{-1} is the multiplicative inverse of $x \in \mathbb{R}$ if and only if

$$x \cdot x^{-1} = x^{-1} \cdot x = 1 \quad .$$

The statement of the theorem requires that (i) $x = \aleph_{\mathcal{X}} + b$, (ii) $0 < \mathcal{X} < 1$, and (iii) $b \in \mathbb{R}_0$. Axiom 5.2.5 grants that multiplication is distributive over addition so the definition of the multiplicative inverse requires

$$\big(\aleph_{\mathcal{X}} + b\big)x^{-1} = \aleph_{(\mathcal{X}x^{-1})} + bx^{-1} = 1 \quad .$$

Equating the big and little parts of this expression, we obtain two requirements

$$\aleph_{(\mathcal{X}x^{-1})} = \aleph_0 \qquad \Longleftrightarrow \qquad \mathcal{X}x^{-1} = 0 \qquad \Longleftrightarrow \qquad x^{-1} = 0 \quad ,$$

and

$$bx^{-1} = 1 \qquad \Longleftrightarrow \qquad x^{-1} = \frac{1}{b} \quad .$$

This contradicts the requirement $b \in \mathbb{R}_0$ and so, therefore, $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ does not have a multiplicative inverse.

Theorem 5.2.16 *All real numbers $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ have an additive inverse.*

Proof. The number x^{-1} is the additive inverse of x if and only if

$$x + x^{-1} = x^{-1} + x = 0 \quad .$$

The statement of the theorem requires that (i) $x = \aleph_{\mathcal{X}} + b$, (ii) $0 < \mathcal{X} < 1$, and (iii) $b \in \mathbb{R}_0$. Assume that x^{-1} has the form $\aleph_{(\mathcal{X}^{-1})} + b^{-1}$. The definition of the additive inverse requires

$$1 = \big(\aleph_{\mathcal{X}} + b\big) + \big(\aleph_{(\mathcal{X}^{-1})} + b^{-1}\big) = \aleph_{(\mathcal{X}+\mathcal{X}^{-1})} + (b + b^{-1}) \quad .$$

Equating the big and little parts of this expression, we obtain two requirements

$$\aleph_{(\mathcal{X}+\mathcal{X}^{-1})} = \aleph_0 \qquad \Longleftrightarrow \qquad \mathcal{X} + \mathcal{X}^{-1} = 0 \qquad \Longleftrightarrow \qquad \mathcal{X}^{-1} = -\mathcal{X} \quad ,$$

and

$$b + b^{-1} = 1 \qquad \Longleftrightarrow \qquad b^{-1} = -b \quad .$$

For every $[\mathcal{X}], [b] \subset C_{\mathbb{Q}}$ there exists a $[-\mathcal{X}], [-b] \subset C_{\mathbb{Q}}$ so, therefore, every $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ has an additive inverse.

Definition 5.2.17 A divisive identity is a number e satisfying $x \div e = x$. The divisive identity element of $\mathbb{R}$ is $1 \in \mathbb{R}_0$.

Theorem 5.2.18 *All real numbers $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ have a non-unique divisive inverse.*

Proof. If x^{-1} is the divisive inverse of x, then $x \div x^{-1} = 1$. By Axiom 5.2.11, any two $x \in \{\mathbb{R}_0^{\mathcal{X}}\}$ having equal big parts are mutual divisive inverses.

§5.3 Limit Considerations Regarding the Arithmetic Axioms

We have not directly defined $\widehat{\infty}$ with the limit definition of infinity. Instead, we have defined infinity hat to have the same absolute value as infinity so that they are both the unincluded endpoint of the interval $[0, \mathcal{I})$ where $\mathcal{I} \notin \mathbb{R}$ has the property that it is larger than any real number. Although we began with the notion of $\mathbf{AB} \equiv [0, \infty]$, by the introduction of the semantic conventions regarding geometric and algebraic infinity, we would now say that ∞ cannot be included as an endpoint so that $[0, \widehat{\infty}) = [0, \infty)$ but, informally, $[0, \widehat{\infty}] \neq [0, \infty]$ because the latter closed interval contradicts the notion of infinite geometric extent. In general, we have only introduced this convention as a thinking device and there is no reason to directly forbid the usual extended real interval $\overline{\mathbb{R}} = [-\infty, \infty]$. Rather, we have only shown that it is better to write $\overline{\mathbb{R}} = [-\widehat{\infty}, \widehat{\infty}]$ because it doesn't suggest the non-existence of the neighborhood of infinity.

So, although we have not defined $\widehat{\infty}$ directly with the limit definition of ∞, having instead deduced its existence from the geometric invariance of line segments under permutations of the labels of their endpoints, it remains that the magnitude of $\widehat{\infty}$ is given by the limit definition. Since the identity of real numbers is identically their magnitude, and it is only two alternative sets of arithmetic axioms which separate ∞ and $\widehat{\infty}$, in this section we will study the compliance of the limit definition of infinity with the arithmetic axioms.

Example 5.3.1 Although the limit definition of ∞ is said to be its identical definition, we cannot always substitute the limit definition of infinity to directly compute all expressions involving geometric infinity. Consider the use of Definition 2.2.2 to write

$$\infty - \infty = \left(\lim_{x\to 0}\frac{1}{x}\right) - \left(\lim_{y\to 0}\frac{1}{y}\right) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{y-x}{xy} \quad .$$

Generally, this limit does not exist because we obtain different results on the lines $y = x$ and $y = 2x$. Presently, however, there is only one possible line: the real number line. By making the substitution for the limit definition, we find, therefore, that

$$\infty - \infty = \left(\lim_{x\to 0}\frac{1}{x}\right) - \left(\lim_{x\to 0}\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0 \quad .$$

This contradicts Axiom 2.2.3 which gives

$$\infty - \infty = \text{undefined} \quad .$$

To the contrary, if we examine $\widehat{\infty} - \widehat{\infty}$ under the ansatz that this expression may be computed with the limit definition, then we find

$$\widehat{\infty} - \widehat{\infty} = \left(\lim_{x\to 0}\frac{1}{x}\right) - \left(\lim_{x\to 0}\frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x}\right) = \lim_{x\to 0} 0 = 0 \quad .$$

This is exactly what is given in Axiom 4.4.13 so the ansatz is borne out. Other identities such as Axiom 5.1.1 giving $\frac{b}{\infty} = 0$ for $b \in \mathbb{R}_0$ *do* follow directly from the limit definition of geometric infinity. We have for $b \in \mathbb{R}_0$

$$\frac{b}{\infty} = \frac{b}{\lim\limits_{x\to 0}\frac{1}{x}} = \lim_{x\to 0}\frac{b}{\frac{1}{x}} = \lim_{x\to 0} xb = 0 \quad .$$

Remark 5.3.2 Example 5.3.1 has demonstrated that although ∞ is directly defined with the limit definition of infinity, we cannot always use that definition to simplify ∞'s expressions. Also, we can use it *sometimes* to simplify the expressions of $\widehat{\infty}$. In the present section, as in Example 4.3.10, we will take the hat on $\widehat{\infty}$ as a constraint on the freedom of algebraic manipulations involving the limit expression. Particularly, the non-absorptivity of $\widehat{\infty}$ allows us to combine limit expressions but forbids us moving any scalars into the limit expressions. The main purpose of Section 5.3 is to demonstrate cases of the validity of the ansatz that sometimes we can correctly compute expressions involving $\widehat{\infty}$ by making the direct substitution with the limit definition.

Theorem 5.3.3 *The property of Axioms 5.2.3 and 5.2.5 giving for* $a, b \in \mathbb{R}_0^+$

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad ,$$

follows from the limit definition of infinity.

Proof. Proof follows from direct substitution of the limit definition of infinity (Definition 2.2.2.) We have

$$\begin{aligned}\big(\widehat{\infty}-b\big)-\big(\widehat{\infty}-a\big)&=\left[\left(\lim_{x\to0}\frac{1}{x}\right)-b\right]-\left[\left(\lim_{x\to0}\frac{1}{x}\right)-a\right]\\&=\lim_{x\to0}\left(\frac{1}{x}-b-\frac{1}{x}+a\right)\\&=\lim_{x\to0}\big(-b+a\big)\\&=a-b\quad.\end{aligned}$$

Theorem 5.3.4 *The property of Axioms 5.2.3 and 5.2.5 giving for* $a,b\in\mathbb{R}_0$ *and* $0<\min(\mathcal{X},\mathcal{Y})\leq\max(\mathcal{X},\mathcal{Y})<1$

$$\big(\aleph_{\mathcal{X}}+b\big)-\big(\aleph_{\mathcal{Y}}+a\big)=\aleph_{(\mathcal{X}-\mathcal{Y})}-a+b\quad,$$

follows from the limit definition of infinity.

Proof. Proof follows from direct substitution of the limit definition of infinity (Definition 2.2.2.) We have

$$\begin{aligned}\big(\aleph_{\mathcal{X}}+b\big)-\big(\aleph_{\mathcal{Y}}+a\big)&=\big(\mathcal{X}\,\widehat{\infty}-b\big)-\big(\mathcal{Y}\,\widehat{\infty}-a\big)\\&=\left[\mathcal{X}\left(\lim_{x\to0}\frac{1}{x}\right)+b\right]-\left[\mathcal{Y}\left(\lim_{x\to0}\frac{1}{x}\right)+a\right]\\&=\big(\mathcal{X}-\mathcal{Y}\big)\left(\lim_{x\to0}\frac{1}{x}\right)-a+b\\&=\big(\mathcal{X}-\mathcal{Y}\big)\widehat{\infty}-a+b\\&=\aleph_{(\mathcal{X}-\mathcal{Y})}-a+b\quad.\end{aligned}$$

Remark 5.3.5 Theorem 5.3.4 requires clarification because we might have written

$$\begin{aligned}\big(\aleph_{\mathcal{X}}+b\big)-\big(\aleph_{\mathcal{Y}}+a\big)&=\big(\mathcal{X}\,\widehat{\infty}+b\big)-\big(\mathcal{Y}\,\widehat{\infty}+a\big)\\&=\left[\left(\lim_{x\to0}\frac{\mathcal{X}}{x}\right)+b\right]-\left[\left(\lim_{x\to0}\frac{\mathcal{Y}}{x}\right)+a\right]\\&=\left(\lim_{x\to0}\frac{\mathcal{X}-\mathcal{Y}}{x}\right)-a+b\\&=\widehat{\infty}-a+b\quad.\end{aligned}$$

Since $\widehat{\infty}=\aleph_1$, this would necessarily be a contradiction. The condition $0<\min(\mathcal{X},\mathcal{Y})\leq\max(\mathcal{X},\mathcal{Y})\leq1$ forbids $\mathcal{X}-\mathcal{Y}=1$. In the above algebraic manipulation, we have given at the second step

$$\aleph_{\mathcal{X}}=\mathcal{X}\,\widehat{\infty}=\lim_{x\to0}\frac{\mathcal{X}}{x}\quad.$$

This contradicts Definition 4.3.7 requiring that $\widehat{\infty}$ does not have absorptive properties. Such a property is explicitly bestowed to the limit definition of infinity when we move the scalar $\mathcal{X}$ into the limit expression. Therefore, it is implicit in the axioms that scalar multipliers of $\widehat{\infty}$ must not be transferred by multiplicative association into the limit expression when substituting the limit definition of algebraic infinity $\widehat{\infty}$. In practice, this has little to no relevance because arithmetic follows from the arithmetic axioms rather than the limit definition of infinity. The purpose of the present section, rather, is to show that at least many of the axioms may be derived from the limit definition, and that ***the present axiomatic framework is very strong*** because many of its axioms are directly provable when we assume the usual associativities, commutativities, and distributivities constrained by the rules of non-absorptivity.

Theorem 5.3.6 *The property of Axioms 5.2.3 and 5.2.5 giving for* $a, b \in \mathbb{R}_0^+$ *and* $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) \leq 1$

$$\big(\aleph_{\mathcal{X}} + b\big) \cdot a = \aleph_{(\mathcal{X}a)} + ba \quad ,$$

follows from the limit definition of infinity.

Proof. Proof follows from direct substitution of the limit definition of infinity. We have

$$\begin{aligned}
\big(\aleph_{\mathcal{X}} + b\big) \cdot a &= \big(\mathcal{X}\widehat{\infty} + b\big) \cdot a \\
&= \left[\mathcal{X}\left(\lim_{x\to 0}\frac{1}{x}\right) - b\right] \cdot a \\
&= \mathcal{X}a\left(\lim_{x\to 0}\frac{1}{x}\right) + ba \\
&= \mathcal{X}a\widehat{\infty} + ba \\
&= \aleph_{(\mathcal{X}a)} + ba \quad .
\end{aligned}$$

Theorem 5.3.7 *The property of Axioms 5.2.3 and 5.2.5 giving for* $a, b \in \mathbb{R}_0$ *and* $0 < \min(\mathcal{X}, \mathcal{Y}) \leq \max(\mathcal{X}, \mathcal{Y}) < 1$

$$\big(\aleph_{\mathcal{X}} - b\big) \cdot \big(\aleph_{\mathcal{Y}} - a\big) = \aleph_{(\aleph_{(\mathcal{X}\mathcal{Y})} + a\mathcal{X} + b\mathcal{Y})} + ba \quad ,$$

follows from the limit definition of infinity.

Proof. Proof of the present theorem follows from direct substitution of the limit definition of infinity. We have

$$\begin{aligned}
\big(\aleph_{\mathcal{X}} - b\big)\big(\aleph_{\mathcal{Y}} - a\big) &= \big(\mathcal{X}\widehat{\infty} - b\big)\big(\mathcal{Y}\widehat{\infty} - a\big) \\
&= \left[\mathcal{X}\left(\lim_{x\to 0}\frac{1}{x}\right) - b\right]\left[\mathcal{Y}\left(\lim_{x\to 0}\frac{1}{x}\right) - a\right]
\end{aligned}$$

$$= \mathcal{X}\mathcal{Y}\left(\lim_{x\to0}\frac{1}{x}\right)^2 - a\left(\lim_{x\to0}\frac{1}{x}\right) - b\left(\lim_{x\to0}\frac{1}{x}\right) + ba$$

If we wrote here

$$\widehat{\infty}\cdot\widehat{\infty} = \left(\lim_{x\to0}\frac{1}{x}\right)^2 = \lim_{x\to0}\frac{1}{x^2} = \widehat{\infty} \quad ,$$

then that would not exactly violate Definition 4.3.7 because it shows infinity absorbing itself while Definition 4.3.5 gives the the multiplicative absorptive property in terms of a composition between $\widehat{\infty}$ and $x \in \mathbb{R}$. However, moving the exponent into the limit violates Definition 4.3.11 requiring that

$$\widehat{\infty}\cdot\widehat{\infty} = \widehat{\infty}\cdot\aleph_1 = \aleph_{\widehat{\infty}} \neq \aleph_1 = \widehat{\infty} \quad .$$

Therefore, we finish the proof as

$$\begin{aligned}
\big(\aleph_{\mathcal{X}} - b\big)\big(\aleph_{\mathcal{Y}} - a\big)a &= \mathcal{X}\mathcal{Y}\left(\lim_{x\to0}\frac{1}{x}\right)\aleph_1 - a\aleph_1 - b\aleph_1 + ba \\
&= \aleph_{\mathcal{X}\mathcal{Y}\left(\lim_{x\to0}\frac{1}{x}\right)} - \aleph_a - \aleph_b + ba \\
&= \aleph_{\mathcal{X}\mathcal{Y}}\cdot\infty - \aleph_{a+b} + ba \\
&= \aleph_{(\aleph_{(\mathcal{X}\mathcal{Y})}+a\mathcal{X}+b\mathcal{Y})} + ba \quad .
\end{aligned}$$

Theorem 5.3.8 *The property of Axiom 5.2.11 giving for $a, b \in \mathbb{R}_0$ and $0 < \min(\mathcal{X},\mathcal{Y}) \leq \max(\mathcal{X},\mathcal{Y}) < 1$*

$$\frac{\aleph_{\mathcal{X}} + b}{\widehat{\infty}} = \mathcal{X} \quad ,$$

follows from the limit definition of infinity.

Proof. We will use the property that $\mathcal{X} \in \mathbb{R}_0$ to allow us move it out of the quotient, as per Axiom 5.2.10. We have

$$\frac{\aleph_{\mathcal{X}} + b}{\widehat{\infty}} = \frac{\mathcal{X}\left(\lim_{x\to0}\frac{1}{x}\right)}{\lim_{x\to0}\frac{1}{x}} + \frac{b}{\lim_{x\to0}\frac{1}{x}} = \mathcal{X}\frac{\lim_{x\to0}\frac{1}{x}}{\lim_{x\to0}\frac{1}{x}} = \mathcal{X}\lim_{x\to0}1 = \mathcal{X} \quad .$$

Remark 5.3.9 The property of Axiom 5.2.11 giving for $a, b \in \mathbb{R}_0$ and $0 < \min(\mathcal{X},\mathcal{Y}) \leq \max(\mathcal{X},\mathcal{Y}) < 1$

$$\frac{a}{\aleph_{\mathcal{X}} + b} = 0 \quad ,$$

does not follow from the limit definition of infinity. If we wrote

$$\frac{a}{\aleph_{\mathcal{X}}+b}=\frac{a}{\mathcal{X}\left(\lim\limits_{x\to 0}\frac{1}{x}\right)+b}=\frac{a}{\mathcal{X}}\cdot\frac{1}{\left(\lim\limits_{x\to 0}\frac{1}{x}\right)+\frac{b}{\mathcal{X}}}\quad ,$$

then we would have way to evaluate the quotient without bringing the denominator's $\frac{b}{\mathcal{X}}$ term into the limit expression. If we did, then the expected zero output would follow directly but moving that term into the limit expression is not allowed because doing so would give $\widehat{\infty}$ an additive absorptive property.

Theorem 5.3.10 *The quotient of any $\mathbb{R}_0$ number divided by any number with a non-vanishing big part is identically zero.*

Proof. Suppose $x, b \in \mathbb{R}_0^+$ and $0 < \mathcal{X} < 1$ and that

$$\frac{x}{\aleph_{\mathcal{X}}+b}=z\quad .$$

Axiom 5.2.10 allows us to take $x \in \mathbb{R}_0$ out of the quotient so we may write

$$\frac{1}{\aleph_{\mathcal{X}}+b}=\frac{z}{x}\quad .$$

The quotient is only well defined for $z = 0$.

Theorem 5.3.11 *Quotients of the form $\mathbb{R}_0^{\mathcal{X}} \div \mathbb{R}_0^{\mathcal{Y}}$ are always equal to $\frac{\mathcal{X}}{\mathcal{Y}}$.*

Proof. By Theorem 5.3.10, we have

$$\frac{\aleph_{\mathcal{X}}+b}{\aleph_{\mathcal{Y}}+a}=\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}}+a}+\frac{b}{\aleph_{\mathcal{Y}}+a}=\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}}+a}\quad .$$

If $a = 0$, then

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}}}=\frac{\mathcal{X}\lim\limits_{x\to 0}\frac{1}{x}}{\mathcal{Y}\lim\limits_{x\to 0}\frac{1}{x}}=\frac{\mathcal{X}}{\mathcal{Y}}\cdot\frac{\lim\limits_{x\to 0}\frac{1}{x}}{\lim\limits_{x\to 0}\frac{1}{x}}=\frac{\mathcal{X}}{\mathcal{Y}}\cdot\lim_{x\to 0}1=\frac{\mathcal{X}}{\mathcal{Y}}\quad .$$

To prove the present theorem in the general case of a, we will demonstrate a contradiction. Suppose $c \neq \frac{\mathcal{X}}{\mathcal{Y}}$ and that

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{\mathcal{Y}}+a}=c\quad .$$

Further suppose that $\mathcal{X} < \mathcal{Y}$ so that we may assume $0 < c < 1$. Then c has a multiplicative inverse and

$$\frac{\aleph_{\mathcal{X}}}{\aleph_{(c\mathcal{Y})}+ca}=1\quad .$$

Then

$$\lim_{a\to 0}\frac{\aleph_{\mathcal{X}}}{\aleph_{(c\mathcal{Y})}+ca}=\frac{\aleph_{\mathcal{X}}}{\aleph_{(c\mathcal{Y})}}=\frac{\mathcal{X}}{c\mathcal{Y}}=1\qquad\Longleftrightarrow\qquad c=\frac{\mathcal{X}}{\mathcal{Y}}\quad .$$

It follows that the small part of the denominator does not contribute to the quotient. The case of $\mathcal{X}>\mathcal{Y}$ follows from the case of $a=0$. The theorem is proven.

Example 5.3.12 This example demonstrates that the associativity of multiplication and division for $\mathbb{R}_0$ numbers such as c. Consider the expression

$$c\cdot\frac{\aleph_{\mathcal{X}}+b}{\aleph_{\mathcal{Y}}+a}=c\cdot\frac{\mathcal{X}}{\mathcal{Y}}=\frac{c\mathcal{X}}{\mathcal{Y}}\quad .$$

If we move c into the quotient and perform the multiplication before the division, then

$$c\cdot\frac{\aleph_{\mathcal{X}}+b}{\aleph_{\mathcal{Y}}+a}=\frac{c\cdot\big(\aleph_{\mathcal{X}}+b\big)}{\aleph_{\mathcal{Y}}+a}=\frac{\aleph_{(c\mathcal{X})}+cb}{\aleph_{\mathcal{Y}}+a}=\frac{c\mathcal{X}}{\mathcal{Y}}\quad ,$$

demonstrates that the operation remains well-defined with the special associative operations for $\mathbb{R}_0$

Example 5.3.13 This example treats the negative exponent inverse notation. We have

$$\frac{x}{\aleph_{\mathcal{X}}+b}=0\qquad\not\Longrightarrow\qquad\frac{\aleph_{\mathcal{X}}+b}{x}=\frac{1}{0}\quad .$$

The usual "invert and multiply" rule for dividing by fractions relies on an assumed associativity between multiplication and division, and so it cannot be used in certain cases of numbers with non-vanishing big parts. We have

$$\frac{\aleph_{\mathcal{X}}+b}{x}=\aleph_{\left(\frac{\mathcal{X}}{x}\right)}+\frac{b}{x}\quad ,\qquad\text{and}\qquad\left(\frac{x}{\aleph_{\mathcal{X}}+b}\right)^{-1}=\frac{1}{\left(\dfrac{x}{\aleph_{\mathcal{X}}+b}\right)}=\text{undefined}\quad .$$

§5.4 Field Axioms

In earlier work on the neighborhood of infinity [8], we studied exclusively the maximal neighborhood of infinity using the symbol $\widehat{\mathbb{R}}$ to refer to what we have labeled $\mathbb{R}_0^1$ in the present conventions. To build numbers of the form $x=\widehat{\infty}-b$ in the set $\widehat{\mathbb{R}}\sim\mathbb{R}_0^1$, it was only required to suppress the additive absorption of $\widehat{\infty}$. The remaining multiplicative absorption resulted in certain (undesirable?) mathematical artifacts which are presently eliminated by the total suppression of all absorptive properties for $\widehat{\infty}$. Here, we will list those artifacts which are cured in the present conventions and later in this section we will examine that which remains yet still disagrees with the field axioms.

If $\widehat{\infty}$ retains multiplicative absorption, then for $n, b \in \mathbb{N}$ we have

$$n\big(\widehat{\infty}-b\big) \leq \big(\widehat{\infty}-b\big) \quad .$$

This ordering relation is not supported by the geometric notion of multiplication. The product of any positive number x multiplied by a natural number should be greater than or equal to x. Another cured artifact is observed in the sums of numbers in the maximal neighborhood of infinity. Even without multiplicative absorption, the geometric notion of the difference is preserved with

$$\big(\widehat{\infty}-b\big)-\big(\widehat{\infty}-a\big)=a-b \quad ,$$

but the notion of the sum is not. With multiplicative absorption in place, adding two $\mathbb{R}_0^1$ numbers yields

$$\big(\widehat{\infty}-b\big)+\big(\widehat{\infty}-a\big)=2\widehat{\infty}-\big(b+a\big)=\widehat{\infty}-\big(b+a\big) \quad . \tag{5.1}$$

The geometric notion of addition would require that the sum of two numbers just less than infinity would not be another number just less than infinity. This issue is cured in the present convention with the implicit transfinite ordering $\aleph_{0.9}+\aleph_{0.9}=\aleph_{1.8}\gg\aleph_1$.

The most undesirable artifact (most significant problem?) with allowing $\widehat{\infty}$ to retain multiplicative absorption is the loss of additive associativity. Subtracting $(\widehat{\infty}-c)$ from both sides of Equation (5.1) yields

$$\Big[\big(\widehat{\infty}-b\big)+\big(\widehat{\infty}-a\big)\Big]-\big(\widehat{\infty}-c\big)=\Big[\widehat{\infty}-\big(b+a\big)\Big]-\big(\widehat{\infty}-c\big) \quad .$$

Assuming the associative property of addition, we may arrange the LHS brackets to write

$$\begin{aligned}\big(\widehat{\infty}-b\big)+\Big[\big(\widehat{\infty}-a\big)-\big(\widehat{\infty}-c\big)\Big]&=\Big[\widehat{\infty}-\big(b+a\big)\Big]-\big(\widehat{\infty}-c\big)\\ \widehat{\infty}+\Big[c-\big(b+a\big)\Big]&=c-\big(b+a\big) \quad .\end{aligned}$$

Subtracting the $\mathbb{R}_0$ part from both sides yields the plain contradiction $\widehat{\infty}=0$. This was avoided, originally, by revoking additive associativity in Reference [8]. In the present conventions, we avoid this undesirable result by taking away the multiplicative absorption of infinity hat.

While it is permissible, in principle, to have notions of addition and multiplication which are not inherently geometric, it is highly undesirable for basic arithmetic if addition is not associative. Indeed, it is tantamount to arbitrary to say, "$\widehat{\infty}$ has one kind of absorption but not the other," so the present convention is better because it gives operations which are inherently geometric *and* wherein addition has the highly desirable associative property. Now that we have reviewed the issues that were cleared up, in the present section we will give a common statement of the field axioms together with the ordering and completeness axioms, and then we will make comparisons to the given arithmetic axioms.

Definition 5.4.1 A field is a set S together with the addition and multiplication operators which satisfies the addition and multiplication axioms for fields: Axioms 5.4.2 and 5.4.4.

Axiom 5.4.2 The addition axioms for fields are

- (A1) S is closed under addition: If $x, y \in S$, then $x + y \in S$.
- (A2) Addition is commutative: If $x, y \in S$, then $x + y = y + x$.
- (A3) Addition is associative: If $x, y, z \in S$, then $(x+y)+z = x+(y+z)$.
- (A4) There exists an additive identity element 0 in S: If $x \in S$, then $x + 0 = x$.
- (A5) Every $x \in S$ has an additive inverse: If $x \in S$, then there exists $-x \in S$ such that $x + (-x) = 0$.

Remark 5.4.3 The arithmetic axioms do not exhibit (A1) but they do exhibit (A2)-(A5).

Axiom 5.4.4 The multiplication axioms for fields are

- (M1) S is closed under multiplication: If $x, y \in S$, then $x \cdot y \in S$.
- (M2) Multiplication is commutative: If $x, y \in S$, then $x \cdot y = y \cdot x$.
- (M3) Multiplication is associative: If $x, y, z \in S$, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- (M4) There exists a multiplicative identity element $1 \neq 0$ in S: If $x \in S$, then $x \cdot 1 = x$.
- (M5) If $x \in S$ and $x \neq 0$, then x has a multiplicative inverse: If $x \in S$, then there exists $x^{-1} \in S$ such that $x \cdot x^{-1} = 1$.

Remark 5.4.5 The arithmetic axioms preserve (M2)-(M4) but both of (M1) and (M5) are lost. The loss of (M5) was proven in Theorem 5.2.15.

Definition 5.4.6 An ordered field is a field F together with a relation $<$ which satisfies the field ordering axioms: Axiom 5.4.7.

Axiom 5.4.7 The field ordering axioms are

- (O1) Elements of F have trichotomy: If $x, y \in F$, then one and only one of the following is true: (i) $x < y$, (ii) $x = y$, or (iii) $x > y$.
- (O2) The $<$ relation is transitive: If $x, y, z \in F$, then $x < y$ and $y < z$ together imply $x < z$.

- (O3) If $x, y, z \in F$, then $x < y$ implies $x + z < y + z$.
- (O4) If $x, y, z \in F$, and if $z > 0$, then $x < y$ implies $x \cdot z < y \cdot z$.

It is understood that $x < y$ means $y > x$.

Theorem 5.4.8 *For any $\mathcal{X} > 0$, $\aleph_{\mathcal{X}}$ is an upper bound of $\mathbb{R}_0$.*

Proof. An upper bound of a set is greater than or equal to every element of that set. Suppose

$$X, Y \in \mathbf{AB} \quad , \quad x \in \mathbb{R}_0 \quad , \quad x \in X \quad , \quad \text{and} \qquad \aleph_{\mathcal{X}} \in Y \quad .$$

It follows that

$$\mathcal{D}_{\mathbf{AB}}(AX) = 0 \quad , \quad \text{and} \qquad \mathcal{D}_{\mathbf{AB}}(AY) = \mathcal{X} \quad .$$

By the ordering of $\mathbb{R}$ (Axioms 3.1.13 and 5.2.14), $\aleph_{\mathcal{X}}$ is an upper bound of $\mathbb{R}_0$ whenever $\mathcal{X} > 0$.

Corollary 5.4.9 $\mathbb{N}$ *is bounded from above.*

Proof. If $n \in \mathbb{N}$, then $n \in \mathbb{R}_0$. By Theorem 5.4.8, all $x \in \mathbb{R}_0$ are bounded from above. $\mathbb{N}$ is bounded from above.

Proposition 5.4.10 $\mathbb{R}_0 \subset \mathbb{R}$ does not have a least upper bound $\sup(\mathbb{R}_0) \in \mathbb{R}$. In other words, $\mathbb{R}$ does not have the least upper bound property.

Justification. To invoke a contradiction, suppose $s \in \mathbb{R}$ is a least upper bound of $\mathbb{R}_0$. If $s - 1$ was an upper bound of $\mathbb{R}_0$, then s could not be the least upper bound because $s - 1 < s$. Therefore, $s = \sup(\mathbb{R}_0)$ implies $(s - 1) \in \mathbb{R}_0$. By Axiom 5.2.1, $\mathbb{R}_0$ is closed under addition. It follows that $(s - 1 + 2) \in \mathbb{R}_0$ because $2 \in \mathbb{R}_0$. Since $s + 1 > s$, we obtain a contradiction having shown that there exist elements of $\mathbb{R}_0$ greater than the assumed supremum s.

Definition 5.4.11 The issue described in the justification of Proposition 5.4.10 shall be referred to as "the least upper bound problem."

Remark 5.4.12 Proposition 5.4.10 is usually presented as a theorem and it brings us to one of the most finely nuanced issues in the present treatment of $\mathbb{R}$. This proposition makes a convincing case that $\mathbb{R}_0$ cannot have a supremum in $\mathbb{R}$. However, if $\mathbb{R}_0$ is a subset of the connected interval $(-\widehat{\infty}, \widehat{\infty})$, then it most certainly must have a least upper bound. Otherwise $(-\widehat{\infty}, \widehat{\infty})$ is not connected. We will continue to develop the principles related to whether or not the different open neighborhoods can have suprema in $\mathbb{R}$, and then in

Section 7.5 we will return to the topic of algebraic contradictions related to the suprema required for the connectedness of the interval. If $\mathbb{R}$ is to have the usual topology overall, then it must have the least upper bound property.

§5.5 Compliance of Cauchy Equivalence Classes with the Arithmetic Axioms

In this section, we give the usual definitions for arithmetic operations on Cauchy equivalence classes. We clarify the meanings for the extended case of $[x] \to [X+x] = [\aleph_{\mathcal{X}} + x]$ and then we prove in a few cases that the arithmetic axioms are satisfied by the extended Cauchy equivalence classes $[X+x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \setminus C_{\mathbb{Q}} \implies [X+x] \notin \mathbb{R}_0$. The proofs in this section mostly follow References [11, 12].

Theorem 5.5.1 *Every convergent rational sequence of terms $a_n \in \mathbb{Q}$ is a Cauchy sequence.*

Proof. Per Definition 4.2.2, a sequence $\{a_n\}$ is a Cauchy sequence if and only if

$$\forall \delta \in \mathbb{Q} \quad \exists m, n, N \in \mathbb{N} \quad \text{s.t.} \quad m, n > N \quad \implies \quad \big| a_n - a_m \big| < \delta \quad .$$

By the convergence of $\{a_n\}$, it is granted that there exists some $l \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = l \quad .$$

Convergence then guarantees that

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \implies \quad \big| a_n - l \big| < \frac{\delta}{2} \quad .$$

Then, whenever $n, m > N$, we have

$$\big| a_n - a_m \big| = \big| \big(a_n - l \big) - \big(a_m - l \big) \big| \le \big| a_n - l \big| + \big| a_m - l \big| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad .$$

Therefore, every convergent rational sequence $\{a_n\}$ is a Cauchy sequence. ✐

Definition 5.5.2 If $x, y \in \mathbb{R}$ such that there are two Cauchy equivalence classes $x = [(x_n)]$ and $y = [(y_n)]$, then $x + y = [(x_n + y_n)]$ and $x \cdot y = [(x_n \cdot y_n)]$.

Theorem 5.5.3 *The additive operation for equivalence classes given by Definition 5.5.2 is well-defined.*

Proof. Define four Cauchy equivalence classes $[(a_n)], [(b_n)], [(c_n)]$, and $[(d_n)]$ having the properties

$$[a] = [b] \quad , \quad \text{and} \qquad [c] = [d] \quad ,$$

so that

$$\lim_{n\to\infty}\big(a_n - b_n\big) = 0 \quad , \quad \text{and} \qquad \lim_{n\to\infty}\big(c_n - d_n\big) = 0 \quad .$$

For addition to be proven well-defined, we need to prove that $[(a_n + c_n)] = [(b_n + d_n)]$. This requires

$$[(a_n + c_n)] - [(b_n + d_n)] = 0 \quad .$$

The difference being equal to zero means that for sufficiently large n, and for any $\delta \in \mathbb{R}$, we must have

$$[(a_n + c_n)] - [(b_n + d_n)] = [(a_n - b_n)] - [(c_n - d_n)] < \delta \quad .$$

We will prove this by the same method of Theorem 5.5.1. The limits of $a_n - b_n$ and $c_n - d_n$ approaching zero tell us that

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \qquad \Longrightarrow \qquad \big|a_n - b_n\big| < \frac{\delta}{2} \quad , \quad \big|c_n - d_n\big| < \frac{\delta}{2} \quad .$$

Then, whenever $n, m > N$, we have

$$\Big|\big(a_n - b_n\big) - \big(c_m - d_m\big)\Big| \le \big|a_n - b_n\big| + \big|c_m - d_m\big| < \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad .$$

This proves that $[a+c] = [b+d]$ and that, therefore, addition is a well-defined operation on Cauchy equivalence classes.

Example 5.5.4 This example gives a specific case of Theorem 5.5.3 using numbers in the neighborhood of infinity. Suppose there are four subsets of $C_{\mathbb{Q}}^{\mathbf{AB}}$ with the properties

$$[\aleph_{[\mathcal{X}_1]} + x_1] = [\aleph_{[\mathcal{Y}_1]} + y_1] \quad , \quad \text{and} \qquad [\aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{Y}_2]} + y_2] \quad .$$

Since the big and little parts of equal numbers are equal, we have equality among all the matched pairs of $[x_1], [x_2], [y_1], [y_2], [\mathcal{X}_1], [\mathcal{X}_2], [\mathcal{Y}_1], [\mathcal{Y}_2] \subset C_{\mathbb{Q}}$. If addition is well-defined, then

$$[\aleph_{[\mathcal{X}_1]} + x_1] + [\aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{Y}_1]} + y_1] + [\aleph_{[\mathcal{Y}_2]} + y_2] \quad .$$

Evaluating the left and right sides independently yields

$$[\aleph_{[\mathcal{X}_1]} + x_1] + [\aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{X}_1]} + x_1 + \aleph_{[\mathcal{X}_2]} + x_2] = [\aleph_{[\mathcal{X}_1+\mathcal{X}_2]} + x_1 + x_2] \quad ,$$

and

$$[\aleph_{[\mathcal{Y}_1]} + y_1] + [\aleph_{[\mathcal{Y}_2]} + y_2] = [\aleph_{[\mathcal{Y}_1]} + y_1 + \aleph_{[\mathcal{Y}_2]} + y_2] = [\aleph_{[\mathcal{Y}_1+\mathcal{Y}_2]} + y_1 + y_2] \quad .$$

Considering first the small parts, Definition 5.5.2 gives $[x+y] = [x] + [y]$ so

$$[x_1 + x_2] = [y_1 + y_2] \qquad \Longleftrightarrow \qquad [x_1] + [x_2] = [y_1] + [y_2] \quad .$$

This condition follows from Theorem 5.5.3. Considering the big parts yields

$$[\aleph_{[\mathcal{X}_1+\mathcal{X}_2]}] = [\aleph_{[\mathcal{Y}_1+\mathcal{Y}_2]}] \qquad \Longleftrightarrow \qquad [\mathcal{X}_1] + [\mathcal{X}_2] = [\mathcal{Y}_1] + [\mathcal{Y}_2] \quad .$$

It follows as an obvious corollary of Theorem 5.5.3 that the additive operation is well-defined for numbers in the neighborhood of infinity.

Remark 5.5.5 To prove that the multiplicative operation is well-defined, we will rely on the boundedness of Cauchy sequences. First, we will give the proof of boundedness.

Theorem 5.5.6 *If $\{a_n\}$ is a Cauchy sequence of rationals, then there exists an $M \in \mathbb{R}$ such that $|a_n| < M$ for all $n \in \mathbb{N}$. In other words, every Cauchy sequence of rationals is bounded.*

Proof. Since $\{a_n\}$ is Cauchy, we know there is some sufficiently large $m, n \in \mathbb{N}$ such that

$$\big|a_n - a_m\big| < 1 \quad .$$

If follows for such n that

$$\big|a_{N+1} - a_n\big| < 1 \qquad \Longrightarrow \qquad \big(a_{N+1} - 1\big) < a_n < \big(a_{N+1} + 1\big) \quad .$$

Define M as the greatest element of a set with a natural number of elements

$$M = \max\Big\{\big|a_0\big|, \big|a_1\big|, ..., \big|a_N\big|, \big|a_{N+1} - 1\big|, \big|a_{N+1} + 1\big|\Big\} \quad .$$

Every a_n with $n \leq N$ is in the set, and every a_n with $n > N$ is less than one of the last two elements of the set. Therefore, there exists a bound $M \in \mathbb{R}$ for every rational Cauchy sequence $\{a_n\}$.

Theorem 5.5.7 *The multiplicative operation for equivalence classes given by Definition 5.5.2 is well-defined.*

Proof. Define four Cauchy equivalence classes $[(a_n)], [(b_n)], [(c_n)]$, and $[(d_n)]$ having the properties

$$[a] = [b] \quad , \qquad \text{and} \qquad [c] = [d] \quad ,$$

so that

$$\lim_{n\to\infty}\big(a_n - b_n\big) = 0 \quad , \qquad \text{and} \qquad \lim_{n\to\infty}\big(c_n - d_n\big) = 0 \quad .$$

For multiplication to be proven well-defined, we need to prove that $[(a_n \cdot c_n)] = [(b_n \cdot d_n)]$, or specifically that for sufficiently large n

$$[(a_n \cdot c_n)] - [(b_n \cdot d_n)] < \delta \quad .$$

To that end, insert the additive identity as a difference of cross terms so that

$$\begin{aligned} a_n \cdot c_n - b_n \cdot d_n &= a_n \cdot c_n - b_n \cdot d_n + \big(c_n \cdot b_n - c_n \cdot b_n\big) \\ &= \big(a_n \cdot c_n - c_n \cdot b_n\big) + \big(c_n \cdot b_n - b_n \cdot d_n\big) \\ &= c_n \cdot \big(a_n - b_n\big) + b_n \cdot \big(c_n - d_n\big) \quad . \end{aligned}$$

It follows that

$$\big|a_n \cdot c_n - b_n \cdot d_n\big| \leq \big(\big|c_n\big| \cdot \big|a_n - b_n\big| + \big|b_n\big| \cdot \big|c_n - d_n\big|\big) \quad .$$

By Theorem 5.5.6, there exists bounds $|b_n| \leq B_0$ and $|c_n| \leq C_0$ for any $n \in \mathbb{N}$. Then let $M_0 = B_0 + C_0$ so that

$$\big|a_n \cdot c_n - b_n \cdot d_n\big| < M_0 \left(\big|a_n - b_n\big| + \big|c_n - d_n\big|\right) \quad .$$

Since all four sequences are Cauchy, we have

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \implies \quad \big|a_n - b_n\big| < \frac{\delta}{2M_0} \quad , \quad \big|c_n - d_n\big| < \frac{\delta}{2M_0} \quad .$$

We prove the theorem by writing

$$\big|a_n \cdot c_n - b_n \cdot d_n\big| < M_0 \left(\frac{\delta}{2M_0} + \frac{\delta}{2M_0}\right) = \delta \quad .$$

Remark 5.5.8 Theorem 5.5.6 proves the boundedness of Cauchy sequences of rationals in $C_\mathbb{Q}$ but not the boundedness of all sequences in $C_\mathbb{Q}^{\mathbf{AB}}$. Since numbers with non-zero big parts are represented as ordered pairs of elements of $C_\mathbb{Q}$, it is obvious that such numbers are bounded because each sequence in the pair is bounded. As a consequence of Theorem 5.5.7 which regards general Cauchy equivalence classes and does not restrict to the rationals, it follows that multiplication is well-defined for numbers in the neighborhood of infinity. However, one must carefully note that the boundedness of such products will not always be such that the bound is in $\mathbb{R}$. By the identity $\aleph_\mathcal{X} \cdot \aleph_\mathcal{Y} = \aleph_{\aleph_{(\mathcal{XY})}}$, it is never in $\mathbb{R}$ when $\mathcal{X} > 0$ or $\mathcal{Y} > 0$.

Remark 5.5.9 Assuming the field axioms, Definition 5.5.2 giving $x \cdot y = [(x_n \cdot y_n)]$ is good enough to allow us to prove the arithmetic operations are well-defined. However, we have presently not defined division as multiplication by an inverse so we need to give a definition for the quotient of two Cauchy equivalence classes.

Definition 5.5.10 If $x, y \in \mathbb{R}$ such that there are two Cauchy equivalence classes $x = [(x_n)]$ and $y = [(y_n)]$, then $x \div y = [(x_n \div y_n)]$.

Theorem 5.5.11 *The quotient operation for equivalence classes of rationals given by Definition 5.5.10 is well-defined.*

Proof. Define four Cauchy equivalence classes $[(a_n)], [(b_n)], [(c_n)]$, and $[(d_n)]$ having the properties

$$[a] = [b] \quad , \quad \text{and} \qquad [c] = [d] \quad ,$$

so that

$$\lim_{n\to\infty} \big(a_n - b_n\big) = 0 \quad , \quad \text{and} \qquad \lim_{n\to\infty} \big(c_n - d_n\big) = 0 \quad .$$

For division to be proven well-defined, we need to prove that $[(a_n \div c_n)] = [(b_n \div d_n)]$. Specifically, for sufficiently large n, we must demonstrate

$$[(a_n \div c_n)] - [(b_n \div d_n)] < \delta \quad .$$

To that end, insert the additive identity as a difference of the cross terms so that

$$\begin{aligned} \frac{a_n}{c_n} - \frac{b_n}{d_n} &= \frac{a_n}{c_n} - \frac{b_n}{d_n} + \left(\frac{b_n}{c_n} - \frac{b_n}{c_n}\right) \\ &= \left(\frac{a_n}{c_n} - \frac{b_n}{c_n}\right) + \left(\frac{b_n}{c_n} - \frac{b_n}{d_n}\right) \\ &= \frac{a_n - b_n}{c_n} + \frac{b_n \cdot \big(d_n - c_n\big)}{c_n \cdot d_n} \quad . \end{aligned}$$

It follows that

$$\left|\frac{a_n}{c_n} - \frac{b_n}{d_n}\right| \leq \left(\frac{\big|a_n - b_n\big|}{\big|c_n\big|} + \frac{\big|b_n\big| \cdot \big|c_n - d_n\big|}{\big|c_n\big| \cdot \big|d_n\big|}\right) \quad .$$

By Theorem 5.5.6, there exist bounds $|b_n| \leq B_0$, $|c_n| \leq C_0$ and $|d_n| \leq D_0$ for any $n \in \mathbb{N}$. Since all four sequences are Cauchy, we have

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \qquad \Longrightarrow \qquad \big|a_n - b_n\big| < \frac{C_0\delta}{2} \quad , \quad \big|c_n - d_n\big| < \frac{C_0 D_0 \delta}{2B_0} \quad .$$

We prove the theorem by writing

$$\left|\frac{a_n}{c_n} - \frac{b_n}{d_n}\right| < \left(\frac{\frac{C_0\delta}{2}}{C_0} + \frac{B_0 \frac{C_0 D_0 \delta}{2B_0}}{C_0 D_0}\right) = \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad .$$

Since we have assumed $[a], [b], [c], [d] \subset C_{\mathbb{Q}}$, we have proven the theorem with Axiom 5.2.10 granting associativity among division and multiplication.

Main Theorem 5.5.12 *The quotient operation given by Definition 5.5.10 is well-defined for equivalence classes in* $C_{\mathbb{Q}}^{\mathbf{AB}} \setminus C_{\mathbb{Q}}$.

Proof. Suppose there are four subsets of $C_{\mathbb{Q}}^{\mathbf{AB}}$ with the properties

$$[\aleph_{[\mathcal{A}]} + a] = [\aleph_{[\mathcal{B}]} + b] \quad , \quad \text{and} \qquad [\aleph_{[\mathcal{C}]} + C] = [\aleph_{[\mathcal{D}]} + d] \quad .$$

It follows from the equality of Cauchy sequences that

$$\begin{aligned}\lim_{n\to\infty}\big(\mathcal{A}_n-\mathcal{B}_n\big)&=0\\\lim_{n\to\infty}\big(\mathcal{C}_n-\mathcal{D}_n\big)&=0\\\lim_{n\to\infty}\big(a_n-b_n\big)&=0\\\lim_{n\to\infty}\big(c_n-d_n\big)&=0\quad.\end{aligned}$$

For concision in notation, introduce the symbols

$$\begin{aligned}(A_n)&=(\aleph_{[(\mathcal{A}_n)]}+a_n)\\(B_n)&=(\aleph_{[(\mathcal{B}_n)]}+b_n)\\(C_n)&=(\aleph_{[(\mathcal{C}_n)]}+c_n)\\(D_n)&=(\aleph_{[(\mathcal{D}_n)]}+d_n)\quad.\end{aligned}$$

For division to be proven well-defined, we need to prove that $[(A_n\div C_n)]=[(B_n\div D_n)]$. Specifically, for sufficiently large n, we must demonstrate

$$[(A_n\div C_n)]-[(B_n\div D_n)]<\delta\quad.$$

Following the form of Theorem 5.5.11, we may insert the identity to obtain the inequality

$$\left|\frac{A_n}{C_n}-\frac{B_n}{D_n}\right|\leq\frac{\big|A_n-B_n\big|}{\big|C_n\big|}+\frac{\big|B_n\big|\cdot\big|C_n-D_n\big|}{\big|C_n\big|\cdot\big|D_n\big|}\quad.$$

Here we make the major distinction with Theorem 5.5.11: the bounds on $(A_n),(B_n),(C_n),(D_n)$ are not in $\mathbb{R}_0$ and we must be careful not to allow associativity among multiplication and division when simplifying the expression. Since each of $(A_n),(B_n),(C_n),(D_n)$ are ordered pairs of Cauchy sequences of rationals (Axiom 4.2.17), we know the pairs of sequences are bounded. Let the bounds be defined by

$$\begin{aligned}[(A_n)]&=([\mathcal{A}],[a])\leq(A_0,a_0)\\[(B_n)]&=([\mathcal{B}],[b])\leq(B_0,b_0)\\[(C_n)]&=([\mathcal{C}],[c])\leq(C_0,c_0)\\[(D_n)]&=([\mathcal{D}],[d])\leq(D_0,d_0)\quad,\end{aligned}$$

where the notation implies the ordering of each paired element respectively. It follows that

$$\begin{aligned}\left|\frac{A_n}{C_n}-\frac{B_n}{D_n}\right|&\leq\frac{\big|\aleph_{A_0}+a_0-\aleph_{B_0}-b_0\big|}{\big|\aleph_{C_0}+c_0\big|}+\frac{\big|\aleph_B+b_0\big|\cdot\big|\aleph_{C_0}+c_0-\aleph_{D_0}-d_0\big|}{\big|\aleph_{C_0}+c_0\big|\cdot\big|\aleph_{D_0}+d_0\big|}\\&\leq\frac{\big|\aleph_{(A_0-B_0)}+a_0-b_0\big|}{\big|\aleph_{C_0}+c_0\big|}+\frac{\big|\aleph_B+b_0\big|\cdot\big|\aleph_{(C_0-D_0)}+c_0-d_0\big|}{\big|\aleph_{C_0}+c_0\big|\cdot\big|\aleph_{D_0}+d_0\big|}\end{aligned}$$

$$\leq \frac{\big|A_0 - B_0\big|}{\big|C_0\big|} + \frac{\big|\aleph_{\big(\aleph_{(B_0C_0-B_0D_0)}+B_0c_0-B_0d_0+b_0C_0-b_0D_0\big)} + b_0c_0 - b_0d_0\big|}{\big|\aleph_{\big(\aleph_{(C_0D_0)}+D_0c_0+d_0C_0\big)} + d_0c_0)\big|}$$
$$\leq \frac{\big|A_0 - B_0\big|}{\big|C_0\big|} + \frac{\big|\aleph_{(B_0C_0-B_0D_0)} + B_0c_0 - B_0d_0 + b_0C_0 - b_0D_0\big|}{\big|\aleph_{(C_0D_0)} + D_0c_0 + d_0C_0\big|}$$
$$\leq \frac{\big|A_0 - B_0\big|}{\big|C_0\big|} + \frac{\big|B_0C_0 - B_0D_0\big|}{\big|C_0D_0\big|}$$
$$\leq \frac{\big|A_0 - B_0\big|}{\big|C_0\big|} + \frac{\big|B_0\big| \cdot \big|C_0 - D_0\big|}{\big|C_0\big| \cdot \big|D_0\big|} \quad .$$

Since $A_0, B_0, C_0, D_0 \in \mathbb{R}_0$, this is the same form achieved in Theorem 5.5.11 and we will conclude the proof in the same way. Use the Cauchy property of the respective sequences to write

$$\exists n, N \in \mathbb{N} \quad \text{s.t.} \quad n > N \quad \Longrightarrow \quad \big|A_n - B_n\big| < \frac{C_0\delta}{2} \;\; , \;\; \big|C_n - D_n\big| < \frac{C_0D_0\delta}{2B_0} \quad .$$

We prove the theorem writing

$$\left|\frac{A_n}{C_n} - \frac{B_n}{D_n}\right| < \frac{\frac{C_0\delta}{2}}{C_0} + \frac{B_0\frac{C_0D_0\delta}{2B_0}}{C_0D_0} = \frac{\delta}{2} + \frac{\delta}{2} = \delta \quad .$$

§6 Arithmetic Applications

§6.1 Properties of the Algebraic Fractional Distance Function Revisited

We have defined the algebraic FDF $\mathcal{D}^\dagger_{AB}$ to totally replicate the behavior of the geometric FDF $\mathcal{D}_{AB}$ with the added property that it should allow us to compute numerical quotients of the form $\frac{AX}{AB}$ without requiring a supplemental constraint of the form $AX = cAB$. In verbose notation, we have

$$\mathcal{D}_{\mathbf{AB}} : \mathbf{AB} \to [0,1] \quad , \quad \text{and} \qquad \mathcal{D}^\dagger_{\mathbf{AB}} : \{[0,\widehat{\infty}];x\} \to [0,1] \quad ,$$

so that the algebraic FDF provides more information by taking the line segment and the chart on the line segment whereas the geometric FDF doesn't know about x.

In Section 3.1, we found that neither the algebraic FDF of the first kind nor the second has the analytic form of $\mathcal{D}^\dagger_{AB}$. The second kind was ruled out by Theorem 3.1.19 when we showed that $\mathcal{D}''_{AB}$ is not one-to-one. $\mathcal{D}'_{AB}$ was provisionally eliminated based on an unallowable discontinuity at infinity. Since $\mathcal{D}_{AB}$ is continuous on its domain, $\mathcal{D}^\dagger_{AB}$ is too. In Theorem 3.1.24 specifically, we showed that $\mathcal{D}'_{AB}$ cannot conform to the Cauchy criterion for continuity at infinity because that criterion always fails at infinity. The nature of the failure is that the criterion gives a requirement

$$|x-\infty|<\delta \qquad \Longleftrightarrow \qquad \delta>\infty \ \ .$$

There is no such δ. What is the source of this discrepancy? The source is the additive absorptive property of infinity giving $\infty - x = \infty$. By now, we have shown that the absorptive properties of all infinite elements are not supported by the invariance of line segments under permutations of their endpoints and we have otherwise given an artificial construction $\widehat{\infty}$ which does not have the problematic properties. In this section, we will revisit the continuity and other properties of $\mathcal{D}'_{AB}$. We will show that ***the algebraic FDF of the first kind <u>does</u> satisfy the Cauchy criterion for a limit at infinity***, something which has been considered historically impossible. In the present section, we will also prove Conjecture 3.1.18 wherein it was postulated that $\mathcal{D}'_{AB}$ is injective. Having shown by the end of the present section that there are no obvious discrepancies between $\mathcal{D}^{\dagger}_{AB}$ and $\mathcal{D}'_{AB}$, we will assume that the algebraic FDF of the first kind is identically $\mathcal{D}^{\dagger}_{AB}$.

Main Theorem 6.1.1 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{\mathbf{AB}}(AX)$ converges to a limit $l=1$ at $B\in\mathbf{AB}$.*

<u>Proof.</u> According to the Cauchy definition of the limit of $f(x)$ as x approaches $\widehat{\infty}$, we say that

$$\lim_{x\to\widehat{\infty}} f(x)=l \ \ ,$$

if and only if

$$\forall\varepsilon>0 \quad \exists\delta>0 \quad \text{s.t} \quad \forall x\in D \ \ ,$$

we have

$$0<|x-\widehat{\infty}|<\delta \qquad \Longrightarrow \qquad |f(x)-l|<\varepsilon \ \ .$$

In Theorem 3.1.24, we attempted to show this limit in the approach to geometric infinity $x\to\infty$. At that point, we had to stop because there is no $\delta\in\mathbb{R}$ such that $\infty - x<\delta$. Now we may choose $x\in\mathbb{R}$ with the given arithmetic axioms to obtain, for example,

$$|(\widehat{\infty}-b)-\widehat{\infty}|=b \ \ , \quad \text{or} \qquad |\aleph_{\mathcal{X}}-\widehat{\infty}|=\aleph_{(1-\mathcal{X})} \ \ .$$

Per the ordering axiom (Axiom 5.2.14), either of these can be less than some $\delta\in\mathbb{R}$. This remedies the blockage encountered in Theorem 3.1.24 where we found $\delta\in\mathbb{R}$ implies $\infty - x\not<\delta$. Now we may follow the usual prescription for the Cauchy definition of a limit, even at infinity! To that end, let $\delta=\aleph_{\left(\frac{\varepsilon}{2}\right)}$. Then the Cauchy definition requires that

$$0<|x-\widehat{\infty}|<\aleph_{\left(\frac{\varepsilon}{2}\right)} \ \ , \quad \text{and} \qquad |\mathcal{D}'_{\mathbf{AB}}(AX)-\mathcal{D}'_{\mathbf{AB}}(AB)|<\varepsilon \ \ .$$

First we will evaluate δ expression on the left as

$$\widehat{\infty} - x < \aleph_{\left(\frac{\varepsilon}{2}\right)} \qquad \iff \qquad x > \aleph_{\left(1-\frac{\varepsilon}{2}\right)} \quad .$$

Definition 3.1.9 gives $\mathcal{D}'_{AB}$ as

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} & \text{for} \quad X \neq A, \; X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad ,$$

where

$$\frac{\|AX\|}{\|AB\|} = \frac{\text{len}[a,x]}{\text{len}[a,b]} \quad .$$

Evaluation of the ε expression, therefore, yields

$$\left| \frac{\text{len}[0,x]}{\text{len}[0,\widehat{\infty}]} - 1 \right| = \left| \frac{x}{\widehat{\infty}} - 1 \right| < \left| \frac{\aleph_{\left(1-\frac{\varepsilon}{2}\right)}}{\widehat{\infty}} - 1 \right| = \left| \left(1 - \frac{\varepsilon}{2}\right) - 1 \right| = \left| -\frac{\varepsilon}{2} \right| < \varepsilon \quad .$$

Therefore,

$$\lim_{x \to \widehat{\infty}} \mathcal{D}'_{\mathbf{AB}}(AX) = 1 \quad .$$

This limit demonstrates the continuity of $\mathcal{D}'_{\mathbf{AB}}$ at infinity.

Remark 6.1.2 When defining $\mathcal{D}'_{AB}$ and $\mathcal{D}''_{AB}$ in Section 3.1, we were able to show that $\mathcal{D}''_{AB}$ is not one-to-one but we did not yet have the tools to prove that $\mathcal{D}'_{AB}$ is one-to-one on all real line segments. We conjectured it with Conjecture 3.1.18 and now we will use Lemma 6.1.3 to prove it in Theorem 6.1.4.

Lemma 6.1.3 *For any point $X \equiv \mathscr{X} = [x_1, x_2]$ in a real line segment AB, we have $x_1 \in \mathbb{R}_{\aleph}^{\mathcal{X}_0}$ if and only if $x_2 \in \mathbb{R}_{\aleph}^{\mathcal{X}_0}$.*

Proof. For proof by contradiction, suppose that $x_1 \in \mathbb{R}_{\aleph}^{\mathcal{X}_1}$, $x_2 \in \mathbb{R}_{\aleph}^{\mathcal{X}_2}$, and $\mathcal{X}_1 \neq \mathcal{X}_2$. By Definition 4.1.14, there exist $b_1, b_2 \in \mathbb{R}_{\aleph}^0$ such that

$$x_1 = \aleph_{\mathcal{X}_1} + b_1 \quad , \quad \text{and} \qquad x_2 = \aleph_{\mathcal{X}_2} + b_2 \quad .$$

With the $a \leq b$ condition inherent to the $[a,b]$ interval notation, the algebraic FDF tells us that

$$\min[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{\text{len}[0,x_1]}{\text{len}[0,\infty]} = \frac{x_1}{\infty} = \mathcal{X}_1 \quad ,$$

and

$$\max[\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX)] = \frac{\text{len}[0,x_2]}{\text{len}[0,\infty]} = \frac{x_2}{\infty} = \mathcal{X}_2 \quad .$$

It follows from the identity $\mathcal{D}^{\dagger}_{\mathbf{AB}}(AX) = \mathcal{D}_{\mathbf{AB}}(AX)$ that

$$\min[\mathcal{D}_{\mathbf{AB}}(AX)] = \mathcal{X}_1 \quad , \qquad \text{and} \qquad \max[\mathcal{D}_{\mathbf{AB}}(AX)] = \mathcal{X}_2 \quad .$$

By Definition 3.1.1, $\mathcal{D}_{\mathbf{AB}}(AX)$ is one-to-one which requires

$$\mathcal{X}_1 = \mathcal{X}_2 \quad .$$

This contradicts the assumed condition that $\mathcal{X}_1 \neq \mathcal{X}_2$.

Theorem 6.1.4 *The algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$ is injective (one-to-one) on all real line segments.*

<u>*Proof.*</u> (Proof of Conjecture 3.1.18.) Recall that $\mathcal{D}'_{AB} : AB \to [0,1]$ is

$$\mathcal{D}'_{AB}(AX) = \begin{cases} 1 & \text{for} \quad X = B \\ \dfrac{\|AX\|}{\|AB\|} = \dfrac{\text{len}[a,x]}{\text{len}[a,b]} & \text{for} \quad X \neq A, \; X \neq B \\ 0 & \text{for} \quad X = A \end{cases} \quad .$$

Injectivity requires that

$$\mathcal{D}'_{AB}(AX_1) = \mathcal{D}'_{AB}(AX_2) \qquad \Longleftrightarrow \qquad AX_1 = AX_2 \qquad \Longleftrightarrow \qquad X_1 = X_2 \quad .$$

Even if there is an entire interval of numbers in the algebraic representations of each of X_1 and X_2, we have by Lemma 6.1.3:

$$\min[\mathcal{D}'_{AB}(AX_k)] = \max[\mathcal{D}'_{AB}(AX_k)] = \mathcal{X}_k \quad .$$

This tells us that choosing any $x \in \mathscr{X} \equiv X$ will yield the same $\mathcal{D}'_{AB}(AX)$. Therefore, the injectivity of $\mathcal{D}'_{AB}(AX)$ follows from the injectivity of $\mathcal{D}_{AB}(AX)$ through the constraint

$$\mathcal{D}'_{AB}(AX) = \mathcal{D}_{AB}(AX) \quad .$$

Conjecture 6.1.5 The algebraic fractional distance function $\mathcal{D}^{\dagger}_{AB}$ is an algebraic fractional distance function of the first kind $\mathcal{D}'_{AB}$.

§6.2 Some Theorems for Real Numbers in the Neighborhood of Infinity

In Section 3.3, we listed four coarse bins of length as distinct modes in which a line segment might have a many-to-one or one-to-one relationship between its points and the numbers in their algebraic representations. The bins were

- $L \in \mathbb{R}_0$

- $L \in \mathbb{R}^0_\aleph \setminus \mathbb{R}_0$
- $L \in \mathbb{R}^{\mathcal{X}}_\aleph \cup \mathbb{R}^1_\aleph$ (Recall that $0 < \mathcal{X} < 1$ is implicit in the absence of explicit statements to the contrary.)
- $L = \widehat{\infty}$

In Theorems 3.3.1 and 3.3.2, we were able to prove the cases $L \in \mathbb{R}_0$ and $L = \widehat{\infty}$ respectively but we did not yet have sufficient tools to easily demonstrate the cases of $L \in \mathbb{R}^0_\aleph \setminus \mathbb{R}_0$ and $L \in \mathbb{R}^{\mathcal{X}}_\aleph \cup \mathbb{R}^1_\aleph$. We still have not decided whether or not $\mathbb{R}^0_\aleph \setminus \mathbb{R}_0 = \varnothing$ but, by this point, we have given the tools needed to prove the many-to-one relationship between real numbers and points in a line segment with $L \in \mathbb{R}^{\mathcal{X}}_0 \cup \mathbb{R}^1_0$. This is the third case above modified with a restriction to the natural neighborhoods of $\aleph_{\mathcal{X}}$ rather than the whole neighborhoods. This restriction guarantees $\text{Lit}(L) \in \mathbb{R}_0$. We will give this result in the present section which also contains various and sundry theorems and examples, the most exciting of which is left as a surprise.

Theorem 6.2.1 *If AB is a real line segment with finite length $L \in \mathbb{R}^{\mathcal{X}}_0 \cup \mathbb{R}^1_0$, then no point $X \in AB$ has a unique algebraic representation as one and only one real number.*

Proof. From the statement of the theorem, we have $L = \text{len}(AB) = \aleph_{\mathcal{X}} + b$ with $0 < \mathcal{X} \leq 1$ and $b \in \mathbb{R}_0$. By Definition 2.3.15, every point in a line segment has an algebraic representation

$$X \equiv \mathscr{X} = [x_1, x_2] \quad .$$

It follows that

$$\min[\mathcal{D}^\dagger_{AB}(AX)] = \frac{\text{len}[0, x_1]}{\text{len}[0, \aleph_{\mathcal{X}} + b]} = \frac{x_1}{\aleph_{\mathcal{X}} + b} \quad .$$

Now suppose $x_0 \in \mathbb{R}^+_0$, and $z = x_1 + x_0$ so that $z > x_1$. Then

$$\frac{\text{len}[0, z]}{\text{len}[0, \aleph_{\mathcal{X}} + b]} = \frac{z}{\aleph_{\mathcal{X}} + b} = \frac{x_1 + x_0}{\aleph_{\mathcal{X}} + b} = \frac{x_1}{\aleph_{\mathcal{X}} + b} + \frac{x_0}{\aleph_{\mathcal{X}} + b} \quad .$$

By Axiom 5.2.11, the x_0 term vanishes so we find

$$\frac{\text{len}[0, z]}{\text{len}[0, \aleph_{\mathcal{X}} + b]} = \frac{x_1}{\aleph_{\mathcal{X}} + b} = \min[\mathcal{D}^\dagger_{AB}(AX)] \quad .$$

Invoking the single-valuedness of bijective functions, we find that

$$\min[\mathcal{D}^\dagger_{\mathbf{AB}}(AX)] = \max[\mathcal{D}^\dagger_{\mathbf{AB}}(AX)] = \frac{x_2}{\aleph_{\mathcal{X}} + b} \qquad \Longrightarrow \qquad x_1 < z \leq x_2 \quad .$$

Therefore $x_1 \neq x_2$ and the theorem is proven. ∎

Theorem 6.2.2 *The derivative of $f(x) = \aleph_x$ with respect to x is infinite.*

Proof. The definition of the derivative of $f(x)$ with respect to x is

$$\frac{d}{dx}\,f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad .$$

For $f(x) = \aleph_x$, we have

$$\begin{aligned} \frac{d}{dx}\,\aleph_x &= \lim_{\Delta x \to 0} \frac{\aleph_{(x+\Delta x)} - \aleph_x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\aleph_x + \aleph_{\Delta x} - \aleph_x}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\aleph_{\Delta x} \\ &= \aleph_1 \quad . \end{aligned}$$

Definition 6.2.3 For $0 < \mathcal{X} < 1$, $\mathbb{N}_{\mathcal{X}}$ is a subset of real numbers such that

$$\mathbb{N}_{\mathcal{X}} = \left\{ \aleph_{\mathcal{X}} + w \;\middle|\; w \in \mathbb{W} \right\} \quad ,$$

where the whole numbers are $\mathbb{W} = \{..., -2, -1, 0, 1, 2, ...\}$. The set $\{\mathbb{N}_{\mathcal{X}}\}$ is called the set of all $\mathbb{N}_{\mathcal{X}}$ such that $0 < \mathcal{X} < 1$. Complementing $\mathbb{N}$ in the neighborhood of the origin, define a set

$$\widehat{\mathbb{N}} = \left\{ \widehat{\infty} - n \;\middle|\; n \in \mathbb{N} \right\} \quad ,$$

called natural numbers in the maximal neighborhood of infinity. The set of all extended natural numbers is

$$\mathbb{N}_\infty = \mathbb{N} \cup \{\mathbb{N}_{\mathcal{X}}\} \cup \widehat{\mathbb{N}} \quad .$$

Definition 6.2.4 The function E^x is defined as

$$E^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad ,$$

where the sum is taken to mean all $k \in \mathbb{N}_\infty \cup \{0\}$. This function is called the big exponential function.

Theorem 6.2.5 *For any $x \in \mathbb{R}_0$, the big exponential function is equal to the usual exponential function:*

$$x \in \mathbb{R}_0 \quad \Longrightarrow \quad E^x = e^x \quad .$$

Proof. The usual exponential function is

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}\quad ,$$

with the upper bound on k meaning "as k increases without bound" but also giving an implicit restriction $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. To prove the present theorem, it will suffice to show that all terms vanish for $k\notin\mathbb{N}$. We have

$$\begin{aligned}E^x&=\sum_{k\in\mathbb{N}_0}\frac{x^k}{k!}+\sum_{k\in\mathbb{N}_{\mathcal{X}_1}}\frac{x^k}{k!}+\sum_{k\in\mathbb{N}_{\mathcal{X}_2}}\frac{x^k}{k!}+...\\&=e^x+\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{x^{\left(\aleph_{\mathcal{X}_1}+k\right)}}{\left(\aleph_{\mathcal{X}_1}+k\right)!}+\sum_{\substack{k=1\\k\in\mathbb{N}}}^{\infty}\frac{x^{\left(\aleph_{\mathcal{X}_1}-k\right)}}{\left(\aleph_{\mathcal{X}_1}-k\right)!}+\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{x^{\left(\aleph_{\mathcal{X}_2}+k\right)}}{\left(\aleph_{\mathcal{X}_2}+k\right)!}+...\quad .\end{aligned}$$

Now it will suffice to show that the sum over $k\in\mathbb{N}_{\mathcal{X}}$ vanishes for any $\mathcal{X}>0$. Observe that

$$\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{x^{(\aleph_{\mathcal{X}}\pm k)}}{\left(\aleph_{\mathcal{X}}\pm k\right)!}=\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{\left(x^{\mathcal{X}}\right)^{\widehat{\infty}}\left(x^{\pm k}\right)}{\left(\aleph_{\mathcal{X}}\pm k\right)!}\quad .$$

To finish the proof, we will separate the three relevant cases of the magnitude of $x^{\mathcal{X}}$.

- ($x^{\mathcal{X}}<1$) Here, the numerator vanishes by Axiom 5.1.7. It follows that

$$\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{\left(x^{\mathcal{X}}\right)^{\widehat{\infty}}\left(x^{\pm k}\right)}{\left(\aleph_{\mathcal{X}}\pm k\right)!}=\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{0\cdot\left(x^{\pm k}\right)}{\left(\aleph_{\mathcal{X}}\pm k\right)!}=0\quad .$$

- ($x^{\mathcal{X}}=1$) Here we have

$$\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{\left(x^{\mathcal{X}}\right)^{\widehat{\infty}}\left(x^{\pm k}\right)}{\left(\aleph_{\mathcal{X}}\pm k\right)!}=\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{\left(x^{\pm k}\right)}{\left(\aleph_{\mathcal{X}}\pm k\right)!}\quad .$$

To evaluate this, take

$$\left(\aleph_{\mathcal{X}}\pm k\right)!=\aleph_{\aleph_{\aleph_{...}}}=\infty\quad .$$

Since this factorial is not equal to $\aleph_1$ and it diverges in $\mathbb{R}$, it must be equal to ∞. We find, therefore, that

$$\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{x^{(\aleph_{\mathcal{X}}\pm k)}}{\left(\aleph_{\mathcal{X}}\pm k\right)!}=\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty}\frac{\left(x^{\pm k}\right)}{\infty}=0\quad .$$

- ($x^{\mathcal{X}} > 1$) If we follow the simple procedure in the previous cases, we obtain

$$\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \frac{\big(x^{\mathcal{X}}\big)^{\widehat{\infty}}\big(x^{\pm k}\big)}{\big(\aleph_{\mathcal{X}} \pm k\big)!} = \sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \frac{\infty\cdot\big(x^{\pm k}\big)}{\infty} = \text{undefined} \quad .$$

The ∞ symbol is shorthand for a limit so we have

$$\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \frac{\infty\cdot\big(x^{\pm k}\big)}{\infty} = \sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \frac{\lim\limits_{x\to0}\dfrac{x^{\pm k}}{x}}{\lim\limits_{x\to0}\dfrac{1}{x}} = \sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \lim_{x\to0}\frac{x^{\pm k-1}}{x^{-1}} = \sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \lim_{x\to0} x^{\pm k} = 0 \quad .$$

We have shown that every term of E^x which is not in e^x vanishes whenever $x\in\mathbb{R}_0$. It can be demonstrated that for any $\mathcal{X}>0$, the factorial $(\aleph_{\mathcal{X}}\pm k)!$ will exceed $\aleph_{\mathcal{Z}}$ for any $\mathcal{Z}<\widehat{\infty}$ so the given value for the factorial is well-motivated and sound in this context. In this proof, we were very careful to respect the order of operations demanded by the non-associativity of multiplication and division.

Alternatively, since the factorial notation appearing in the series expansion is only a shorthand notation, it is demonstrative to show that the terms belonging only to big exponential function vanish as

$$\begin{aligned}\sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \frac{x^{(\aleph_{\mathcal{X}}\pm k)}}{\big(\aleph_{\mathcal{X}}\pm k\big)!} &= \sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} \frac{x}{\big(\aleph_{\mathcal{X}}\pm k\big)}\frac{x}{\big(\aleph_{\mathcal{X}}\pm k-1\big)}\frac{x^{(\aleph_{\mathcal{X}}\pm k-2)}}{\big(\aleph_{\mathcal{X}}\pm k-2\big)!}\\ &= \sum_{\substack{k=0\\k\in\mathbb{N}_0}}^{\infty} 0\cdot0\cdot\ \ldots\left(\frac{x}{3!}\right)\left(\frac{x}{2!}\right)\left(\frac{x}{1!}\right) = 0 \quad .\end{aligned}$$

The evaluation in this alternative manipulation may or may not require associativity among non-associative operations, depending on the underlying construction of the factorial in the exponential function and how it governs the associativity of the operations. If this second pathway is disallowed because it relies on forbidden associativity, then this theorem is proven by the former method above. In any case, the theorem is proven. ☙

Example 6.2.6 This example gives a good thinking device for understanding limits $n\to\infty$ when n steps in integer multiples. Usually $n\to\infty$ is taken to mean "as the iterator n increases without bound." In this example, we will argue that $n\to\infty$ is better interpreted as meaning "the sum over every $n\in\mathbb{N}_\infty$." Definition 2.2.2 gives two definitions for the ∞ symbol, one of which is

$$\lim_{n\to\infty}\sum_{k=1}^{n} k = \infty \quad .$$

The $n \to \infty$ limit of the partial sums is taken to mean "as n increases without bound" without a self-referential presupposition of the number defined by the limit. Axiom 5.2.1 grants the closure of $\mathbb{R}_0$ under its operations so the partial sums will always be another $\mathbb{R}_0$ number for any $n \in \mathbb{N}$. For any $\mathcal{X} > 0$, it follows that the sum will be less than $\aleph_{\mathcal{X}}$ but the statement "as n increases without bound" induces the notion of the non-convergence of the partial sums. In turn, this allows us to think of the sum as exceeding $\aleph_{\mathcal{X}}$ but it may more plainly demonstrate the notion of non-convergence when we take $n \to \infty$ to mean the sum over all $n \in \mathbb{N}_\infty$. In that case, the partial sums will eventually have individual terms greater than $\aleph_{\mathcal{X}}$ for any $0 < \mathcal{X} < 1$. It is immediately obvious that $\aleph_{\mathcal{X}}$ cannot be an upper bound on the partial sums over $n \in \mathbb{N}_\infty$. The big part of the partial sums will easily exceed $\aleph_1 = \widehat{\infty}$. Taking $m \in \mathbb{N}$, observe that the $n \in \mathbb{N}_\infty$ convention gives

$$\lim_{n\to\infty} \sum_{k=1}^{n} k > \lim_{n\to m} \sum_{k=1}^{n} \big(\aleph_{(m^{-1})} + k \big) > m\aleph_{(m^{-1})} = \aleph_1 \quad .$$

Now it is plainly obvious that the limit of the partial sums diverges in $\mathbb{R}$. Certainly, it is obvious that the partial sums diverge in either case but it may be more obvious when $n \in \mathbb{N}_\infty$. When n is said to increase without bound and is also taken as $n \in \mathbb{N}$, then there is an intuitive hiccup seeing that the sequence of the sums should diverge when every element in the sequence of partial sums is less than any $\aleph_{\mathcal{X}} \in \{\mathbb{R}^{\mathcal{X}}\}$. Instead, it is better to think of the $n \to \infty$ notation as meaning the sum over all $n \in \mathbb{N}_\infty$.

This example has demonstrated the utility of $\mathbb{N}_\infty$ as a thinking device and it also makes a distinction between the two formulae

$$\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty \quad , \quad \text{and} \qquad \lim_{n\to\infty} \sum_{k=1}^{n} k = \infty \quad .$$

In the partial sums definition, and under the $\mathbb{N}_\infty$ convention, the distinction between geometric infinity and algebraic infinity is suggested as

$$\widehat{\infty} = \aleph_1 \quad , \quad \text{and} \qquad \infty = \aleph_\infty \quad .$$

This convention would require a significant revision of the entire text to accommodate $|\aleph_1| \neq |\aleph_\infty|$ but we point out the possibility of the alternative convention with a nod toward future inquiry. Note, however, that the present convention for either definition of ∞ is preserved with Definition 2.2.1: both sums diverge in $\mathbb{R}$ and we cannot differentiate $\aleph_1$ from $\aleph_\infty$ without first making a transfinite analytic continuation. This continuation is surely something to be explored because it is the longitudinal continuation of $\mathbb{R}$ beyond its endpoints perfectly dual to the famous transverse continuation of $\mathbb{R}$ onto $\mathbb{C}$. Where the latter has yielded so much fruit in the history of mathematics, the former ought to bear some fruit as well.

Main Theorem 6.2.7 *If (i) ABC is a right triangle such that $\angle ABC = \frac{\pi}{2}$, (ii) $\|AB\| = \aleph_{\mathcal{X}} + x$, (iii) $\|BC\| = \aleph_{\mathcal{Y}} + y$, and (iv) $\|AB\| \neq c_0\|BC\|$, and if the Pythagorean theorem is phrased as*

$$\|AC\| = \sqrt{\|AB\|^2 + \|BC\|^2} \quad , \quad \textit{with} \qquad \|AC\| = len(AC) \quad ,$$

then

$$len(AC) \notin \mathbb{R}^0_{\aleph} \cup \{\mathbb{R}^{\mathcal{X}}_{\aleph}\} \cup \mathbb{R}^1_{\aleph} \quad .$$

Proof. The squared lengths of the legs are

$$\|AB\|^2 = \big(\aleph_{\mathcal{X}} + x\big)^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2)} + 2x\mathcal{X}\right)} + x^2 \quad ,$$

and

$$\|BC\|^2 = \big(\aleph_{\mathcal{Y}} + y\big)^2 = \aleph_{\left(\aleph_{(\mathcal{Y}^2)} + 2y\mathcal{Y}\right)} + y^2 \quad .$$

If we directly state the Pythagorean theorem in terms of the lengths, then we find

$$\|AC\|^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2 + \mathcal{Y}^2)} + 2(x\mathcal{X} + y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Assuming $\|AC\| = \aleph_{\mathcal{A}} + a$, we find

$$\aleph_{\left(\aleph_{(\mathcal{A}^2)} + 2a\mathcal{A}\right)} + a^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2 + \mathcal{Y}^2)} + 2(x\mathcal{X} + y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Setting the big parts equal yields

$$\aleph_{(\mathcal{A}^2)} + 2a\mathcal{A} = \aleph_{(\mathcal{X}^2 + \mathcal{Y}^2)} + 2(x\mathcal{X} + y\mathcal{Y}) \quad ,$$

which still has separable big and little parts. Doing the maximum possible separation of all the big and little parts yields

$$\begin{aligned} \mathcal{A}^2 &= \mathcal{X}^2 + \mathcal{Y}^2 \\ a\mathcal{A} &= x\mathcal{X} + y\mathcal{Y} \\ a^2 &= x^2 + y^2 \quad . \end{aligned}$$

Here we have three inconsistent equations in two variables a and $\mathcal{A}$. No real-valued length $\|AC\|$ squared will satisfy the Pythagorean theorem as stated. The theorem is proven.

Definition 6.2.8 A number is a complex number $z \in \mathbb{C}$ if and only if

$$z = x + iy \quad , \quad \text{and} \qquad x, y \in \mathbb{R} \quad .$$

Theorem 6.2.9 *If we assign an algebraic representation to the hypotenuse $AC \equiv z \in \mathbb{C}$ rather than the $AC \equiv \|AC\| \in \mathbb{R}$ disallowed by Main Theorem 6.2.7, then the Pythagorean identity is satisfied by $AC^2 \equiv \bar{z}z$.*

Proof. Given two legs, we want to find the hypotenuse through the Pythagorean theorem. We assume that the legs are real line segments so that

$$AB^2 \equiv \|AB\|^2 \quad , \quad \text{and} \qquad BC^2 \equiv \|BC\|^2 \quad .$$

If we take $\|AB\|, \|BC\| \in \mathbb{C}$ such that $\text{Im}(\|AB\|) = \text{Im}(\|BC\|) = 0$, then each is its own complex conjugate and the quantity squared retains its usual meaning. The geometric identity

$$AC^2 = AB^2 + BC^2 \quad ,$$

needs an algebraic interpretation if we are to do trigonometry. The present theorem concerns the "squared," exponent 2 operation being identified as multiplication by the complex conjugate in the sense that the inner product of a 1D vector $\vec{z} \in \mathbb{C}^1$ with itself is $\vec{z}^2 = \langle z|z\rangle = \bar{z}z$. The vector space axioms are known to be satisfied in $\mathbb{C} = \mathbb{C}^1$ so it is only an irrelevant matter of notation whether we specify a complex number z or a 1D complex vector $\vec{z}$. However, the satisfaction of the Pythagorean identity relies critically on the multiplicative "line segment squared" operation being identified as algebraic multiplication by the complex conjugate

$$AC^2 \equiv \langle AC|AC\rangle \quad .$$

Since the legs are taken as real, the algebraic representation of each is its own complex conjugate. As in the previous theorem, we find

$$\big\langle AB\big|AB\big\rangle = \big(\aleph_{\mathcal{X}} + x\big)^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2)}+2x\mathcal{X}\right)} + x^2 \quad ,$$

and

$$\big\langle BC\big|BC\big\rangle = \big(\aleph_{\mathcal{Y}} + y\big)^2 = \aleph_{\left(\aleph_{(\mathcal{Y}^2)}+2y\mathcal{Y}\right)} + y^2 \quad .$$

The present statement of the Pythagorean theorem is

$$\big\langle AC\big|AC\big\rangle = \aleph_{\left(\aleph_{(\mathcal{X}^2+\mathcal{Y}^2)}+2(x\mathcal{X}+y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Let $z \in \mathbb{C}$ be such that (i) it conforms to Definition 6.2.8, (ii) $AC \equiv z = |AC\rangle$, and (iii)

$$z = \aleph_{(\mathcal{X}\pm i\mathcal{Y})} + x \pm iy = \big(\aleph_{\mathcal{X}} + x\big) + i\big(\aleph_{\mathcal{Y}} + y\big) \quad .$$

We have

$$\begin{aligned}
\bar{z}z = \big\langle AC\big|AC\big\rangle &= \big[\aleph_{(\mathcal{X}\pm i\mathcal{Y})} + (x \pm iy)\big]^*\big[\aleph_{(\mathcal{X}\pm i\mathcal{Y})} + (x \pm iy)\big] \\
&= \big[\aleph_{(\mathcal{X}\mp i\mathcal{Y})} + (x \mp iy)\big]\big[\aleph_{(\mathcal{X}\pm i\mathcal{Y})} + (x \pm iy)\big] \\
&= \aleph_{(\mathcal{X}\mp i\mathcal{Y})\aleph_{(\mathcal{X}\pm i\mathcal{Y})}} + \aleph_{(\mathcal{X}\mp i\mathcal{Y})}(x \pm iy) \\
&\qquad\qquad + \aleph_{(\mathcal{X}\pm i\mathcal{Y})}(x \mp iy) + (x \mp iy)(x \pm iy) \\
&= \aleph_{\left(\aleph_{(\mathcal{X}^2+\mathcal{Y}^2)}\right)} + \aleph_{(x\mathcal{X}\pm iy\mathcal{X}\mp ix\mathcal{Y}+y\mathcal{Y})} \\
&\qquad\qquad + \aleph_{(x\mathcal{X}\mp iy\mathcal{X}\pm ix\mathcal{Y}+y\mathcal{Y})} + x^2 + y^2
\end{aligned}$$

$$= \aleph_{\left(\aleph_{(\mathcal{X}^2+\mathcal{Y}^2)}+2(x\mathcal{X}+y\mathcal{Y})\right)} + x^2 + y^2 \quad .$$

Therefore,

$$\big\langle AC\big|AC\big\rangle = \big\langle AB\big|AB\big\rangle + \big\langle BC\big|BC\big\rangle \quad ,$$

and the Pythagorean theorem is satisfied as stated. This proves the theorem.

Corollary 6.2.10 *If we assign an algebraic representation to the hypotenuse $AC \equiv \vec{x} \in \mathbb{R}^2$ rather than the $AC \equiv \|AC\| \in \mathbb{R}$ disallowed by Main Theorem 6.2.7, then the Pythagorean identity is satisfied by $AC^2 \equiv \vec{x}\cdot\vec{x}$.*

Proof. This corollary follows from Theorem 6.2.9 in the way that everything in $\mathbb{C}$ has two equivalent vector space representations in $\mathbb{C}^1$ and $\mathbb{R}^2$. Let $\vec{x} \in \mathbb{R}^2$ be a vector in the Cartesian plane equipped as a vector space. We have three real vectors defining ABC in $\mathbb{R}^2$:

$$\vec{AB} = (\aleph_{\mathcal{X}}+x, 0) \quad , \quad \vec{BC} = (0, \aleph_{\mathcal{Y}}+y) \quad , \quad \text{and} \qquad \vec{AC} = (\aleph_{\mathcal{X}}+x, \aleph_{\mathcal{Y}}+y) \quad .$$

The Pythagorean theorem yields

$$\vec{AC}\cdot\vec{AC} = \vec{AB}\cdot\vec{AB} + \vec{BC}\cdot\vec{BC} \quad .$$

Again we find

$$\vec{AB}\cdot\vec{AB} = \big(\aleph_{\mathcal{X}}+x\big)^2 = \aleph_{\left(\aleph_{(\mathcal{X}^2)}+2x\mathcal{X}\right)} + x^2 \quad ,$$

and

$$\vec{BC}\cdot\vec{BC} = \big(\aleph_{\mathcal{Y}}+y\big)^2 = \aleph_{\left(\aleph_{(\mathcal{Y}^2)}+2y\mathcal{Y}\right)} + y^2 \quad .$$

Checking the given form of $\vec{AC} \in \mathbb{R}^2$, we find

$$\begin{aligned}\vec{AC}\cdot\vec{AC} &= (\aleph_{\mathcal{X}}+x, \aleph_{\mathcal{Y}}+y)\cdot(\aleph_{\mathcal{X}}+x, \aleph_{\mathcal{Y}}+y) \\ &= \vec{AB}\cdot\vec{AB} + \vec{BC}\cdot\vec{BC} \quad .\end{aligned}$$

The Pythagorean identity is satisfied with an algebraic representation of the hypotenuse AC such that $AC \equiv \vec{AC} \in \mathbb{R}^2$. The theorem is proven.

Example 6.2.11 If a right triangle has two equal legs $AB = BC$, then the hypotenuse AC should be such that $AC = \sqrt{2}AB$. Since this is a case of $\|AB\| = c_0\|BC\|$ which was not considered in Main Theorem 6.2.7 we will include it for completeness. We have two equal legs $AB = BC$ such that

$$\sqrt{2}\|AB\| = \sqrt{2}\|BC\| = \aleph_{\sqrt{2}\mathcal{X}} + \sqrt{2}x \quad .$$

We square it to check the Pythagorean theorem and find

$$\big(\aleph_{\sqrt{2}\mathcal{X}} + \sqrt{2}x\big)^2 = \aleph_{\left(\sqrt{2}\mathcal{X}\aleph_{\sqrt{2}\mathcal{X}}\right)} + 2\aleph_{\sqrt{2}\mathcal{X}}\sqrt{2}x + 2x^2$$

$$= \aleph_{\aleph_{(2\mathcal{X}^2)}} + \aleph_{4x\mathcal{X}} + 2x^2$$
$$= \aleph_{\left(\aleph_{2\mathcal{X}^2+4x\mathcal{X}}\right)} + 2x^2$$

Indeed, we find the expected result that the hypotenuse is real-valued and scaled by $\sqrt{2}$ when the legs are equal. It is good that the geometry of a small triangle remains intact even when it is uniformly resized to have a characteristic scale in the neighborhood of infinity. However, what would happen if we rescaled one leg and not the other? Would it suddenly gain an imaginary part? In the complex algebraic representation of the hypotenuse $AC \equiv z = \aleph_{(\mathcal{X}\pm i\mathcal{Y})} + x \pm iy$, as in Theorem 6.2.9, we have randomly chosen $BC \equiv ([\mathcal{Y}], [y])$ as governing the imaginary part. However, if the two legs are real, then why should $([\mathcal{Y}], [y])$ govern the imaginary part of z and not $AB \equiv ([\mathcal{X}], [x])$? Having raised these issues, we leave them to future work on 2D planar Euclidean geometry. After one more related example, we will return to the present considerations regarding the 1D geometry of a straight line (which is not as simple as might be assumed.)

Example 6.2.12 This example demonstrates a ramification of Main Theorem 6.2.7 for the ordinary notions of trigonometry. Consider a right triangle ABC such that $\angle ABC = \frac{\pi}{2}$. Suppose $\|AB\| = \aleph_{\mathcal{X}} + x$ and $\|BC\| = \aleph_{\mathcal{Y}} + y$. Let $\alpha = \angle CAB$ such that $0 < \alpha < \frac{\pi}{2}$. Ordinary notions of trigonometry suggest

$$\|AC\|\sin(\alpha) = \aleph_{\mathcal{Y}} + y \quad , \qquad \text{and} \qquad \|AC\|\cos(\alpha) = \aleph_{\mathcal{X}} + x \quad . \tag{6.1}$$

It follows that

$$\|AC\| = \aleph_{\left(\frac{\mathcal{Y}}{\sin(\alpha)}\right)} + \frac{y}{\sin(\alpha)} \quad , \qquad \text{and} \qquad \|AC\| = \aleph_{\left(\frac{\mathcal{X}}{\cos(\alpha)}\right)} + \frac{x}{\cos(\alpha)}$$

Equating the big and little parts yields

$$\tan(\alpha) = \frac{\mathcal{Y}}{\mathcal{X}} \quad , \qquad \text{and} \qquad \tan(\alpha) = \frac{y}{x} \quad .$$

This is a contradiction for every case in which $\frac{\mathcal{Y}}{\mathcal{X}} \neq \frac{y}{x}$. This is a perfectly consistent result; if the trigonometry functions in Equation (6.1) are real-valued, and each RHS is, then the equality cannot hold when $\text{Im}(\|AC\|) \neq 0$. We have shown in Example 6.2.11 that the hypotenuse is real-valued for the case of $\|AB\| = c_0\|BC\|$ with $c_0 = 1$, and the trigonometry functions should work as usual for any $c_0 \in \mathbb{R}$ because that will enforce the equal relative scale $\frac{\mathcal{Y}}{\mathcal{X}} = \frac{y}{x}$ for the ratios of the big and little parts of the lengths of the legs.

Remark 6.2.13 In leaving the real line and going onto the plane, we have exceeded the scope of this analysis. Other than a $\mathbb{C}$ application in Section 8 to demonstrate the negation of the Riemann hypothesis, we will not continue to exceed the confines of $\mathbb{R}$. Even given the solution to that very famous problem in Section 8, however, it is the opinion of this writer that the principle demonstrated in Main Theorem 6.2.7 is certainly the most important result given herein. It cleanly demonstrates that the extension $L \in \mathbb{R}_0 \to L \in \mathbb{R}_\infty$ is not the trivial exercise that might be intuitively assumed. Among the two valid interpretations given for the Pythagorean identity (Theorem 6.2.9 and Corollary 6.2.10) the $z \in \mathbb{C}$ representation of the length of the hypotenuse is more relevant than $\vec{x} \in \mathbb{R}^2$ because z is a 1D scalar number whose real part is a cut in the real number line. In other words, z is a cut in the real number line added to a cut in the imaginary number line. Since cuts in the real number line are known to have both zero and non-zero imaginary parts, meaning some real numbers are the real parts of complex numbers with non-zero imaginary parts, $z \in \mathbb{C}$ is far more germane to the standard analysis of $\mathbb{R}$ than is $\vec{x} \in \mathbb{R}^2$. Vector structure in vector analysis requires an entire axiomatic framework for vector arithmetic but all of the arithmetic for $z \in \mathbb{C}$ can be derived easily if i is added to the arithmetic axioms.

Theorem 6.2.14 *A real number x with non-vanishing big part is not the product of any real number with itself, i.e.:*

$$\nexists\, z \in \mathbb{R} \quad s.t. \quad z^2 \in \{\mathbb{R}_\aleph^\mathcal{X}\} \cup \mathbb{R}_\aleph^1 \quad .$$

In other words, $x \in \{\mathbb{R}_\aleph^\mathcal{X}\} \cup \mathbb{R}_\aleph^1$ does not have a real-valued square root.

Proof. Let there be two real numbers

$$z = \aleph_\mathcal{Z} + a \quad , \qquad \text{and} \qquad x = \aleph_\mathcal{X} + b \quad .$$

Assume $z^2 = x$ so that

$$\big(\aleph_\mathcal{Z} + a\big)^2 = \aleph_\mathcal{X} + b \quad . \tag{6.2}$$

We have

$$\big(\aleph_\mathcal{Z} + a\big)^2 = \aleph_{\left(\aleph_{(\mathcal{Z}^2)} + 2a\mathcal{Z}\right)} + z^2 \quad ,$$

so we should set the big and little parts of the left and right sides of Equation (6.2) equal to each other. This gives

$$\aleph_{(\mathcal{Z}^2)} + 2a\mathcal{Z} = \mathcal{X} \quad , \qquad \text{and} \qquad z^2 = b \quad .$$

The former constraint equation gives $\mathcal{Z} = 0$ because the RHS has zero big part. It follows that $\mathcal{X} = 0$. This is a contradiction because we have already selected $\aleph_\mathcal{X}$ as the non-zero big part of x.

Example 6.2.15 Consider the limit

$$\lim_{b\to\infty} \aleph_\mathcal{X} - b = l \quad .$$

It remains to be clarified precisely what is meant by the notation $b \to \infty$ because we should have options for at least two distinct behaviors. For example, one might wish to define

$$\lim_{b\to\infty} \aleph_{\mathcal{X}} - b = -\infty \quad , \qquad \text{and} \qquad \lim_{b\to\widehat{\infty}} \aleph_{\mathcal{X}} - b = \aleph_{\mathcal{X}} - \widehat{\infty} = -\aleph_{(1-\mathcal{X})} \quad ,$$

where $b \to \widehat{\infty}$ means that b approaches $\widehat{\infty}$ while $b \to \infty$ would mean that b increases without bound—even including transfinite numbers larger than $\widehat{\infty}$—such that b approaches some geometric infinity whose absolute value is in some sense greater than that of algebraic infinity. We will not make such definitions here because the requisite formal definitions for $x > \widehat{\infty}$ are out of scope. However, simply based on the absorption or non-absorption of ∞ and $\widehat{\infty}$ respectively, the limits given in this example should be presumed correct.

§6.3 The Archimedes Property of Real Numbers

While there are many ways to state the Archimedes property of real numbers with symbolic logic, the modern establishment has adopted a standard statement

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad .$$

For this statement to accurately characterize the property as it appeared in the first edition Greek language copy of Euclid's *Elements*, it must depend on an unstated axiom that every real number is less than some natural number. Without that axiom, the statement is wrong and there is no other word than "wrong" by which it should be described. In this section, we will consult the original text in *The Elements* [1]. We will use the original text to prove absolutely that the above symbolic statement is not the Archimedes property of real numbers given so famously by Euclid in antiquity. For the above statement to agree with that which was given by Euclid in Greek, one must first take the axiom that every real number is less than some natural number. Without a statement or implicit acknowledgment of such an axiom, the above Latin symbolic statement is *wrongly* called the Archimedes property of real numbers.

Definition 6.3.1 The statement of the Archimedes property which appears in Euclid's tome *The Elements*, and which was attributed by Archimedes to his predecessor Eudoxus, *and* which must be taken as *the* definitive statement of the Archimedes property of real numbers, appears as Definition 4 in Book 5 of *The Elements*. The original Greek is translated as follows [1].

> "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another."

Remark 6.3.2 As it appears in *The Elements*, the straightforward mathematical statement of the property would be

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \quad .$$

There is no mention of multiplication by a positive integer $n \in \mathbb{N}$. To prove that the Archimedes property of real numbers does not implicitly restrict the multiplier to $n \in \mathbb{N}$, we will examine the context of the original text.

Definition 6.3.3 In Reference [1], Fitzpatrick translates Book 5, Definitions 1 through 5 as follows.

1. A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.

2. And the greater is a multiple of the lesser whenever it is measured by the lesser.

3. A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.

4. (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.

5. Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and third both exceed, are both equal to, or are both less than, equal multiples of the second and fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.

Remark 6.3.4 Though we may prove directly from Euclid's own words that the multiplier in the Archimedes property is not defined as a natural number, Fitzpatrick gives footnotes qualifying his translations of Euclid's original Greek. These footnotes support the wrongness of the supposition that Euclid meant to imply that the multiplier in must always be a natural number. We will list the footnotes here for thoroughness though we will not rely on them in Theorem 6.3.5. The footnotes are as follows.

1. In other words, α is said to be a part of β if $\beta = m\alpha$.

2. (*No footnote given.*)

3. In modern notation, the ratio of two magnitudes, α and β, is denoted $\alpha : \beta$.

4. In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n.

5. In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\alpha > n\beta$ whenever $m\gamma > n\delta$, $m\alpha = n\beta$ whenever $m\gamma = n\delta$, and $m\alpha < n\beta$ whenever $m\gamma < n\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, *etc.*, are irrational.

Footnote 5 makes it exceedingly obvious that the multipliers are "all m and n" in $\mathbb{R}$.

Theorem 6.3.5 *The statement*

$$\forall x, y \in \mathbb{R} \quad s.t. \quad x < y \quad \exists n \in \mathbb{N} \quad s.t. \quad nx > y \quad .$$

is not a proper statement of the Archimedes property of real numbers as given in antiquity.

Proof. It follows from Book 5, Definition 5, of Euclid's original text that if $y \in \mathbb{R}$ is a multiple of $z \in \mathbb{R}$, then there exists some "multiplier" $x \in \mathbb{R}$ such that $xy = z$. To prove the present theorem by contradiction, assume that Euclid meant to restrict the multiplier in his definitions as $n \in \mathbb{N}$, and then consider Definition 2:

> "And the greater is a multiple of the lesser whenever it is measured by the lesser."

Suppose $y = 2$ and $z = 3$ so that, among the two numbers, z is the greater. If the multiplier by which z is to be measured by y is restricted to $n \in \mathbb{N}$ rather than $x \in \mathbb{R}$, then z cannot be measured by y. This is ***an affront to reason***, firstly, and it directly contradicts Definition 1:

> "A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater."

It is self-evidently true that $3 > 2$ so for 2 to be a part of 3 means it must measure the greater. "Measure" is defined by Definition 2 in terms of multiples which are thence defined in terms of multiplication. For 2 be a part of 3 in the sense of Definition 1, we must do multiplication with a multiplier $x = 1.5 \notin \mathbb{N}$. This proves the theorem.

Remark 6.3.6 In Book 7, Definition 2, Euclid defines "numbers" as natural numbers but what are today called real numbers are instead the "magnitudes" described in Book 5. Euclid in no way implied that the multiplier in Definition 4 should be taken strictly as $n \in \mathbb{N}$ and, so, neither was Euclid of the opinion that Archimedes meant to do so in his own earlier paraphrasing of Eudoxus.

Example 6.3.7 This example demonstrates that if one presupposes the non-existence of real numbers greater than any natural number, taking it purely as an unproven axiomatic definition, one which violates the contrary proof of the existence of such numbers given in Main Theorem 3.2.6, then the statement

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad nx > y \quad ,$$

does adequately encapsulate the Archimedes property of antiquity. Proof of this statement is given by Rudin in Reference [11] as follows.

> "Let A be the set of all nx, where n runs through the positive integers. If [*the symbolic statement given above in the present example*] were false, then y would be an upper bound of A. But then A has a least upper bound in $\mathbb{R}$. Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A. Hence $\alpha - x < mx$ for some positive integer m. But then $\alpha < (m+1)x \in A$, which is impossible since α is an upper bound of A."

Here Rudin has followed the reasoning of Proposition 5.4.10 in which it was claimed that $\mathbb{R}_0$ cannot have a supremum. We will revisit the issue of this supremum most specifically in Section 7.5.

Remark 6.3.8 If we adopt

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R} \quad \text{s.t.} \quad zx > y \quad .$$

as the definitive statement of the Archimedes property, as in Remark 6.3.2, then we will have taken away the Archimedes property of real numbers from the maximal whole neighborhood of infinity $\mathbb{R}^1_\aleph$. For instance, if

$$\widehat{\infty} - a < \widehat{\infty} - b \quad ,$$

then we cannot multiply the LHS by a number greater than one and have a real-valued product due to the identity $\aleph_{\mathcal{X}} = \mathcal{X} \cdot \widehat{\infty}$. If we multiply by a positive number less than one, call it δ, then

$$\aleph_\delta - \delta a < \widehat{\infty} - a < \widehat{\infty} - b \quad ,$$

does *not* conform to the Archimedean requirement that $\delta(\widehat{\infty} - a) > \widehat{\infty} - b$. If this were to force the ejection of $\mathbb{R}^1_\aleph$ from $\mathbb{R}$ because such numbers were found not to exhibit the Archimedes property, then that would cause a breakdown in Axiom 2.1.7 giving $\mathbb{R} = (-\infty, \infty)$. If we suppose, correctly, that all real numbers obey the Archimedes property, then we might write concisely

$$\mathbb{R}^+ = (0, \infty) \setminus \mathbb{R}^1_\aleph \quad . \tag{6.3}$$

This is highly disfavorable because we lose the perfect geometric infinite line construction that we have sought to preserve by modifying the canonical algebraic construction by equivalence classes. In terms of the topology, Equation (6.3) breaks the usual topology of $\mathbb{R}$ such that its basis is no longer all open subsets $(a, b) \subset (-\infty, \infty)$.

In what manner shall the maximal neighborhood of infinity exhibit the Archimedes property of real numbers? How might we solve this problem? The answer lies in Euclid's original Greek:

> "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another."

In Remark 6.3.2, we have adopted the convention that the multiplier must attach to the lesser part of $x < y$ but no such requirement is given by Euclid's totally symmetric statement. For one to exceed the other upon multiplication allows us to state the property in terms of multiplication of either the greater or the lesser among x and y. In Euclid's own parlance, for one to exceed the other only requires that each is a "part" or "multiple" of the other without specifying a requirement for which is which. Taking careful note of the non-specificity of the ordering relation in Euclid's Definition 4, we will preserve the highly favorable definition $\mathbb{R} = (-\infty, \infty)$ by giving a symbolic statement of the Archimedes property obeyed by $x \in \mathbb{R}^1_\aleph$. At the end of this section, we will give a new, modern statement of the Archimedes property such that its application is greatly simplified. First, we will show that the fractional distance model of $\mathbb{R}$ obeys the symbolically complexified Latin restatement of Euclid's small handful of original Greek words. Once we show that the ancient definition is satisfied, will make a simplifying axiom such that demonstrating the property is simplified.

Definition 6.3.9 The most general statement of the ancient Archimedes property of real numbers is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad .$$

Main Theorem 6.3.10 *The present construction of* $\mathbb{R}$ *is such that every* $x, y \in \mathbb{R}^0_\aleph \cup \{\mathbb{R}^{\mathcal{X}}_\aleph\} \cup \mathbb{R}^1_\aleph$ *exhibit the ancient Archimedes property of real numbers.*

Proof. By Definition 6.3.9, it suffices to demonstrate

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad .$$

We will consider the general forms

$$x = \aleph_{\mathcal{X}} + b \quad , \quad \text{and} \qquad y = \aleph_{\mathcal{Y}} + a \quad ,$$

such that $x \in \mathbb{R}^{\mathcal{X}}_\aleph$ and $y \in \mathbb{R}^{\mathcal{Y}}_\aleph$, and we will assume constraints $x < y$ and $0 \leq \mathcal{X} \leq \mathcal{Y} \leq 1$. Further assume that a and b are constrained appropriately for $\mathcal{X}$ and/or $\mathcal{Y}$ equal to one or zero so that x and y are always in $\mathbb{R}^+$. The starting point for demonstrating the Archimedes property is $x < y$ which we write as

$$\aleph_{\mathcal{X}} + b < \aleph_{\mathcal{Y}} + a \quad .$$

To prove the theorem, we will consider the distinct cases. In each equality listed below, we put $z_1 x$ on the left and $z_2 y$ on the right.

• ($x \in \mathbb{R}_\aleph^0$ and $y \in \mathbb{R}_\aleph^0$) Here, both x and y have vanishing big parts so $x < y$ defines the ordering of the little parts. Choose $z_1 = \aleph_\mathcal{Z} + z$ such that $0 < \mathcal{Z}b < 1$ and choose $z_2 = 1$. Then

$$\big(\aleph_\mathcal{Z} + z\big)b = \aleph_{(\mathcal{Z}b)} + zb \quad > \quad a \ .$$

• ($x \in \mathbb{R}_\aleph^0$ and $y \in \{\mathbb{R}_\aleph^\mathcal{Y}\}$) Here, x has a vanishing big part and y has a non-vanishing big part. Choose $z_1 = \aleph_{\left(\frac{1+\mathcal{Y}}{2b}\right)} + z$ and $z_2 = 1$. Then

$$\left(\aleph_{\left(\frac{1+\mathcal{Y}}{2b}\right)} + z\right) b = \aleph_{\left(\frac{1+\mathcal{Y}}{2}\right)} + zb \quad > \quad \aleph_\mathcal{Y} + a \ .$$

Since $\frac{1+\mathcal{Y}}{2}$ is the average of $\mathcal{Y}$ and 1, it is guaranteed to be a number in the open interval $(\mathcal{Y}, 1)$.

• ($x \in \mathbb{R}_\aleph^0$ and $y \in \mathbb{R}_\aleph^1$) Here, x has a vanishing big part and y has big part $\aleph_1$. Choose $z_1 = \aleph_\mathcal{Z} + z$ and z_2 such that $0 < z_2 < \mathcal{Z}b < 1$. Then

$$\left(\aleph_\mathcal{Z} + z\right) b = \aleph_{(\mathcal{Z}b)} + zb \quad > \quad z_2\big(\aleph_1 + a\big) = \aleph_{(z_2)} + z_2 a \ .$$

• ($x \in \mathbb{R}_\aleph^\mathcal{X}$ and $y \in \mathbb{R}_\aleph^\mathcal{Y}$ such that $\mathcal{X} < \mathcal{Y}$) Here, neither x nor y has a vanishing big part and the big part of x is less than big part of y. Choose $z_1 = \frac{1+\mathcal{Y}}{2\mathcal{X}}$ and $z_2 = 1$. Then

$$\frac{1+\mathcal{Y}}{2\mathcal{X}} \left(\aleph_\mathcal{X} + b\right) = \aleph_{\left(\frac{1+\mathcal{Y}}{2}\right)} + b\frac{1+\mathcal{Y}}{2\mathcal{X}} \quad > \quad \aleph_\mathcal{Y} + a \ .$$

• ($x, y \in \mathbb{R}_\aleph^\mathcal{X}$ such that $\mathcal{X} = \mathcal{Y}$) Here, x and y have equal big parts so it follows from $x < y$ that the little parts are ordered accordingly. Choose $z_1 = z$ such that $\mathcal{X} < z\mathcal{X} < 1$ and $z_2 = 1$. Then

$$z\left(\aleph_\mathcal{X} + b\right) = \aleph_{(z\mathcal{X})} + zb \quad > \quad \aleph_\mathcal{X} + a \ .$$

• ($x \in \mathbb{R}_\aleph^\mathcal{X}$ and $y \in \mathbb{R}_\aleph^1$) Here, x and y have unequal big parts with the big part of y being the greater. Choose $z_1 = 1$ and $z_2 = \frac{\mathcal{X}}{2}$. Then

$$\aleph_\mathcal{X} + b \quad > \quad \frac{\mathcal{X}}{2}\big(\aleph_1 - a\big) = \aleph_{\left(\frac{\mathcal{X}}{2}\right)} - \frac{a\mathcal{X}}{2} \ .$$

• ($x, y \in \mathbb{R}_\aleph^1$) Here, x and y have equal big parts $\aleph_1$ and $x < y$ defines the ordering of the little parts. Choose $z_1 = 1$ and $z_2 = \frac{1}{2}$. Then

$$\aleph_1 + b \quad > \quad \frac{1}{2}\big(\aleph_1 - a\big) = \aleph_{\left(\frac{1}{2}\right)} - \frac{a}{2} \ .$$

We have considered every combination of $x < y$ among the various neighborhoods and shown that they comply uniformly with Definition 6.3.9. The theorem is proven.

Remark 6.3.11 If it were not for the extremal case of $x \in \mathbb{R}^0_\aleph$ and $y \in \mathbb{R}^1_\aleph$, we might have formulated the symbolic statement of the Archimedes property as

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z \in \mathbb{R}^+ \quad \text{s.t.} \quad zx > y \quad \text{or} \quad x > zy \quad .$$

This form is nice because it uses only a single multiplication operation and exactly reflects Fitzpatrick's footnote:

> "In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n."

However, it is not possible to phrase the symbolic statement of the property with only a single multiplier because of the extremal case in which x is in the neighborhood of the origin and y is in the maximal neighborhood of infinity. Even then, Euclid does not precisely require a condition of the form, "multiplication of just one can exceed the other." As it is written:

> "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another."

This statement absolutely allows the two multiplier form given in Definition 6.3.9. This statement is equally well clarified with a similar but slightly different footnote than what Fitzpatrick has given. An alternative footnote explaining the meaning of the property would be the following.

> In other words, α has a ratio with respect to β if $m_1\alpha > n_1\beta$ and $m_2\beta > n_2\alpha$, for some m_1, m_2, n_1, and n_2.

This is exactly what is given in Definition 6.3.9 and it is well consistent with the "ratio of ratios" language seen in Book 5, Definition 5.

In general, we have made a rather large statement

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad , \tag{6.4}$$

of Euclid's few original words. The reasoning behind including the Archimedes property of real numbers as a supplemental constraint on the behavior of cuts in the real number line is that it is supposed to be an elegantly simple statement of a simple behavior. Equation (6.4) is not quite elegant. Therefore, having already independently demonstrated rigorous compliance with Euclid in the absence of a simplifying modern axiom, now we will give a simplifying modern axiom.

Axiom 6.3.12 The Archimedes property of 1D transfinitely continued real numbers is

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad \aleph_n x > y \quad .$$

This axiom defines the implicit transfinite ordering required for $\leq$ to be a relation among real numbers $\mathbb{R}$ and 1D transfinitely continued real numbers $\mathbb{T}$ whose big parts are greater than $\widehat{\infty} = \aleph_1$. As a subset of the 1D transfinitely continued real numbers, the real numbers themselves automatically inherit compliance with the Archimedes property.

Remark 6.3.13 If the real number line ends at infinity, that indicates an endpoint there. Endpoints are associated with $\widehat{\infty}$ when we take the convention that the notion of infinite geometric extent precludes the existence of endpoints at ∞. Therefore, the lack of a terminating point for the line at infinity automatically implies the 1D transfinite continuation of $\mathbb{R} = (-\aleph_1, \aleph_1)$ onto $\mathbb{T} = (-\aleph_\infty, \aleph_\infty)$. If $\mathbb{R}$ didn't continue onto $\mathbb{T}$, then it would end at ∞, a contradiction if ∞ is not allowed to be an endpoint. There is no requirement whatsoever that $\widehat{\infty} = \aleph_1$ is the largest number; it is only required that it is the supremum of the real numbers. Axiom 6.3.12 generates the requisite definition of transfinite ordering such that given $x, y \in \mathbb{R}$ and $x < y$, zx can be greater than y without zx itself being $zx \in \mathbb{R}$. Here we define ordering for $zx \in \mathbb{T}$, and then we use this ordering to satisfy the $zx > y$ condition irreducibly cited in *The Elements*. In the scheme of Axiom 6.3.12, all the bulleted cases of $x < y$ statements in Main Theorem 6.3.10 are replaced with elegantly simple formulae.

§7 The Topology of the Real Number Line

§7.1 Basic Set Properties

In this section, we give some elementary set properties of the natural neighborhoods and begin to approach the logical connection to the whole neighborhoods. Recall that the natural neighborhoods $\mathbb{R}_0^{\mathcal{X}}$ are defined with little part $|b| \in \mathbb{R}_0^0$ and the whole neighborhoods are defined with $|b| \in \mathbb{R}_\aleph^0$.

Lemma 7.1.1 *Every natural neighborhood in* $\{\mathbb{R}_0^{\mathcal{X}}\}$ *is an open set.*

Proof. By Definition 4.1.4, the set of all intermediate natural neighborhoods of infinity is

$$\left\{\mathbb{R}_0^{\mathcal{X}}\right\} = \{\aleph_{\mathcal{X}} + b \mid b \in \mathbb{R}_0, \; 0 < \mathcal{X} < 1\} \quad .$$

A given $\mathbb{R}_0^{\mathcal{X}}$ defined with a particular $\mathcal{X}$ is open if and only if there is a δ-neighborhood of each of its elements such that every element of that neighborhood is also an element of $\mathbb{R}_0^{\mathcal{X}}$. We use the ball function δ-neighborhood as in Definition 4.1.11 rather than Definition 4.1.12 because the elements of $\mathbb{R}_0^{\mathcal{X}}$

are numbers, not points. This theorem is proven with a δ-neighborhood of an arbitrary $x \in \mathbb{R}_0^{\mathcal{X}}$. Defining $b^\pm = b \pm \delta$, we have

$$\text{Ball}(x \in \mathbb{R}_0^{\mathcal{X}}, \delta) = (\aleph_{\mathcal{X}} + b - \delta, \aleph_{\mathcal{X}} + b + \delta) = (\aleph_{\mathcal{X}} + b^-, \aleph_{\mathcal{X}} + b^+) \quad .$$

Axiom 5.2.1 requires that $\mathbb{R}_0$ is closed under the $\pm$ operations so $b, \delta \in \mathbb{R}_0$ implies $b^\pm \in \mathbb{R}_0$. The set $\mathbb{R}_0^{\mathcal{X}}$ is open because

$$(\aleph_{\mathcal{X}} + b^-, \aleph_{\mathcal{X}} + b^+) \subset \mathbb{R}_0^{\mathcal{X}} = \left\{ \aleph_{\mathcal{X}} + b \;\middle|\; b \in \mathbb{R}_0 \right\} \quad .$$

Alternatively, no set in $\{\mathbb{R}_0^{\mathcal{X}}\}$ contains its boundary points so each such set is open. ☙

Theorem 7.1.2 *Given two natural neighborhoods $\mathbb{R}_0^{\mathcal{X}}$ and $\mathbb{R}_0^{\mathcal{Y}}$ with $0 \leq \mathcal{X} < \mathcal{Y} \leq 1$, there exists another natural neighborhood $\mathbb{R}_0^{\mathcal{Z}}$ such that $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$.*

Proof. Consider the interval

$$(\aleph_{\mathcal{X}}, \aleph_{\mathcal{Y}}) \subset \mathbb{R} \quad .$$

By Definition 3.2.1, the number at the center of this interval is

$$\frac{\aleph_{\mathcal{Y}} + \aleph_{\mathcal{X}}}{2} = \aleph_{\left(\frac{\mathcal{Y}+\mathcal{X}}{2}\right)} \quad .$$

We have

$$\mathcal{X} < \frac{\mathcal{Y} + \mathcal{X}}{2} < \mathcal{Y} \quad ,$$

so let $\mathcal{Z} = \frac{\mathcal{Y}+\mathcal{X}}{2}$. Any number $z \in Z \in \mathbf{AB}$ of the form

$$z = \aleph_{\mathcal{Z}} + z_0 \quad , \quad \text{for} \qquad |z_0| \in \mathbb{R}_0 \quad ,$$

will be such that

$$\mathcal{D}_{\mathbf{AB}}(AZ) = \mathcal{Z} \quad .$$

Since $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$, the theorem is proven. ☙

Corollary 7.1.3 *Given two whole neighborhoods $\mathbb{R}_\aleph^{\mathcal{X}}$ and $\mathbb{R}_\aleph^{\mathcal{Y}}$ with $0 \leq \mathcal{X} < \mathcal{Y} \leq 1$, there exists another whole neighborhood $\mathbb{R}_\aleph^{\mathcal{Z}}$ such that $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$.*

Proof. Following the proof of Theorem 7.1.2, we arrive at a number $z \in Z \in \mathbf{AB}$ of the form

$$z = \aleph_{\mathcal{Z}} + z_0 \quad , \quad \text{for} \qquad z_0 \in \mathbb{R}_\aleph^0 \quad ,$$

Even in the whole neighborhood exceeding the natural neighborhood, z_0 has no fractional magnitude with respect to $\mathbf{AB}$. Therefore, the total fractional distance is still completely determined by the big part as

$$\text{Big}(z) = \aleph_{\mathcal{Z}} \quad \Longleftrightarrow \quad \mathcal{D}_{\mathbf{AB}}(AZ) = \mathcal{Z} \quad .$$

Proof follows from $\mathcal{X} < \mathcal{Z} < \mathcal{Y}$, as in Theorem 7.1.2. ☙

Definition 7.1.4 An open set S is disconnected if and only if there exist two open, non-empty sets U and V such that

$$S = U \cup V \quad , \qquad \text{and} \qquad U \cap V = \varnothing \quad .$$

If a set is not disconnected, then it is connected.

Corollary 7.1.5 $\mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_0^{\mathcal{Y}}$ *is a disconnected set for any* $0 \leq \mathcal{X} < \mathcal{Y} \leq 1$.

Proof. An open set is disconnected if it is the union of two disjoint, non-empty open sets. By Lemma 7.1.1, $\mathbb{R}_0^{\mathcal{X}}$ is open, and it is obvious that such sets are non-empty. It follows from Theorem 7.1.2 that they are disjoint, *i.e.*:

$$\mathbb{R}_0^{\mathcal{X}} \cap \mathbb{R}_0^{\mathcal{Y}} = \varnothing \quad .$$

The union $\mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_0^{\mathcal{Y}}$ satisfies the definition of a disconnected set. ∎

Remark 7.1.6 During the development of the intermediate neighborhoods of infinity, we found it useful to separate the $\mathcal{X} = 0$ and $\mathcal{X} = 1$ cases from the intermediate neighborhoods $\{\mathbb{R}_{\aleph}^{\mathcal{X}}\}$. For efficacy of notation, now we will combine all the different neighborhoods into a streamlined, unified notation. The following definitions supplement Definitions 4.1.3 and 4.1.4 to include the cases of $\mathcal{X} = 0$ and $\mathcal{X} = 1$.

Definition 7.1.7 To streamline notation, define

$$\mathbb{R}_{\aleph}^{\cup} = \bigcup_{0 \leq \mathcal{X} \leq 1} \mathbb{R}_{\aleph}^{\mathcal{X}} = \mathbb{R}_{\aleph}^{0} \cup \left\{ \mathbb{R}_{\aleph}^{\mathcal{X}} \right\} \cup \mathbb{R}_{\aleph}^{1}$$
$$\mathbb{R}_{0}^{\cup} = \bigcup_{0 \leq \mathcal{X} \leq 1} \mathbb{R}_{0}^{\mathcal{X}} = \mathbb{R}_{0}^{0} \cup \left\{ \mathbb{R}_{0}^{\mathcal{X}} \right\} \cup \mathbb{R}_{0}^{1} \quad .$$

Definition 7.1.8 The complement of $\mathbb{R}_0^{\mathcal{X}}$ in $\mathbb{R}_{\aleph}^{\mathcal{X}}$ is $\mathbb{R}_C^{\mathcal{X}}$:

$$\mathbb{R}_C^{\mathcal{X}} = \mathbb{R}_{\aleph}^{\mathcal{X}} \setminus \mathbb{R}_0^{\mathcal{X}} \quad .$$

Theorem 7.1.9 *There exist more positive real numbers than are in* $\mathbb{R}_0^{\cup}$. *In other words,*

$$\mathbb{R}^{+} \setminus \mathbb{R}_0^{\cup} \neq \varnothing \quad .$$

Proof. By the definition of an interval, and through Axiom 2.1.7 overtly granting the connectedness of $\mathbb{R} = (-\infty, \infty)$, the interval $\mathbb{R}^{+} = (0, \infty)$ is connected. To prove the present theorem, it will suffice to show that $\mathbb{R}_0^{\cup}$ is disconnected. Disconnection follows from Corollary 7.1.5. ∎

Remark 7.1.10 It was already expected that there may be real numbers not contained in the natural neighborhoods. It was for this reason that we defined distinct whole neighborhoods $\mathbb{R}_{\aleph}^{\mathcal{X}} \supseteq \mathbb{R}_0^{\mathcal{X}}$. In Section 7.4, we will conjecture $\mathbb{R}_C^{\mathcal{X}} = \varnothing$ but first we will prove another result, one far more interesting.

Main Theorem 7.1.11 *There exist more positive real numbers than are in $\mathbb{R}_{\aleph}^{\cup}$. In other words,*

$$\mathbb{R}^+ \setminus \mathbb{R}_{\aleph}^{\cup} \neq \varnothing \quad .$$

Proof. $\mathbb{R}^+$ is a connected interval. $\mathbb{R}_{\aleph}^{\cup} \setminus \{0\}$ is a disjoint union of open subsets of $\mathbb{R}^+$. A connected interval cannot be covered with a disconnected set. The theorem is proven.

§7.2 Cantor-like Sets of Real Numbers

In this section, we will continue to develop the properties of $\mathbb{R}$ by comparing the properties of $\mathbb{R}^+ \setminus \mathbb{R}_{\aleph}^{\cup}$ and $\mathbb{R}^+ \setminus \mathbb{R}_0^{\cup}$ to the well-known properties of Cantor sets.

Definition 7.2.1 Munkres constructs a Cantor set C as follows [13].

> "Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. Let A_1 be the set obtained from A_0 by deleting its 'middle third' $(\frac{1}{3}, \frac{2}{3})$. Let A_2 be the set obtained from A_1 by deleting its 'middle thirds' $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. In general, define A_n by the equation
>
> $$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right) \quad .$$
>
> The intersection
>
> $$C = \bigcap_{n \in \mathbb{Z}^+} A_n \quad ,$$
>
> is called the [*ternary*] Cantor set; it is a subspace of $[0,1]$."

Remark 7.2.2 The interval $[0,1]$ is the image of **AB** under the fractional distance map. This likeness will serve as the basis for the analytical direction of the present section.

Definition 7.2.3 Define two Cantor-like sets

$$\mathbb{F}_0 = [0,\infty] \setminus \mathbb{R}_0^{\cup} \quad , \quad \text{and} \qquad \mathbb{F}_{\aleph} = [0,\infty] \setminus \mathbb{R}_{\aleph}^{\cup} \quad .$$

Corollary 7.2.4 *Neither $\mathbb{F}_0$ nor $\mathbb{F}_{\aleph}$ is the empty set.*

Proof. Proof follows from Theorem 7.1.9 and Main Theorem 7.1.11: there exist more positive real numbers than are in either variant of $\mathbb{R}^\cup$. The interval $[0,\infty]$ is connected by Axiom 2.1.7. A connected set cannot be covered by a disjoint union of its open subsets. (We say $\mathbb{R}_0^0$ and $\mathbb{R}_\aleph^0$ are open sets in the subspace topology of $[0,\infty]$ even though they are not strictly open in $\mathbb{R}$.) The corollary is proven.

Remark 7.2.5 To construct $\mathbb{F}_0$ and $\mathbb{F}_\aleph$, we have subtracted from $[0,\infty]$ the neighborhood of $\aleph_\mathcal{X}$ for every infinite decimal number $0 \leq \mathcal{X} \leq 1$. Whatever remains is a "dust" of some sort. For this reason, we call $\mathbb{F}_0$ and $\mathbb{F}_\aleph$ Cantor-like sets.

Lemma 7.2.6 *$\mathbb{F}_\aleph$ is a subset of $\mathbb{F}_0$ or it is exactly equal to $\mathbb{F}_0$, i.e.: $\mathbb{F}_\aleph \subseteq \mathbb{F}_0$.*

Proof. Proof follows from Definition 7.2.3. $\mathbb{F}_\aleph$ is constructed by deleting open intervals whose lengths are at least as great as those deleted in the construction of $\mathbb{F}_0$. Each variant of deleted interval is centered about $\aleph_\mathcal{X}$. $\mathbb{F}_\aleph \subseteq \mathbb{F}_0$ because $\mathbb{R}_0^\mathcal{X} \subseteq \mathbb{R}_\aleph^\mathcal{X}$. If $\mathbb{R}_C^\mathcal{X} = \varnothing$, then $\mathbb{F}_\aleph = \mathbb{F}_0$ (which is what we will choose in Section 7.4.)

Theorem 7.2.7 *$\mathbb{F}_0$ and $\mathbb{F}_\aleph$ are closed subsets of $[0,\infty]$.*

Proof. A subset $S \subset T$ is closed in T if and only if its complement in T is open. The complements of $\mathbb{F}_0$ and $\mathbb{F}_\aleph$ in $[0,\infty]$ are $\mathbb{R}_0^\cup$ and $\mathbb{R}_\aleph^\cup$ respectively, both of which are disjoint unions of open sets. $\mathbb{F}_0$ and $\mathbb{F}_\aleph$ are closed in $[0,\infty]$.

Remark 7.2.8 When constructing the ternary Cantor set (Definition 7.2.1), the least element of the final result of iterative deletions is zero. By construction, the endpoints of the intervals left after each deletion of a middle third will remain forever so it is already given at the A_1 step that the least number in the parent interval $[0,1]$, which is zero, will be the least element of the resultant Cantor set. When defining $\mathbb{F}$ in either variant, it is not immediately apparent what will be the least element because $0 \in \mathbb{R}_\aleph^0 \supseteq \mathbb{R}_0^0$ is deleted at the first step. Since $\mathbb{F}$ is closed, however, we know it does have a least element.

Definition 7.2.9 The connected elements of $\mathbb{F}_0$ are provisionally labeled $\mathbb{F}_0(n)$ and the connected elements of $\mathbb{F}_\aleph$ are provisionally labeled $\mathbb{F}_\aleph(n)$. The labeling convention in either variant is such that

$$\forall x \in \mathbb{F}(n) \quad \forall y \in \mathbb{F}(m) \quad \text{s.t.} \quad n > m \quad \implies \quad x > y \enspace .$$

Each $\mathbb{F}(n)$ is connected and every two $\mathbb{F}(n), \mathbb{F}(m)$ are disconnected whenever $n \neq m$.

Remark 7.2.10 We have deleted an uncountable infinity of $\mathbb{R}_\aleph^\mathcal{X}$ neighborhoods to construct $\mathbb{F}_\aleph$. The elements of $\mathbb{F}_\aleph$ separate these neighborhoods so the number of disconnected elements $\mathbb{F}_\aleph$ must be uncountably infinite. Such elements cannot be enumerated with $n \in \mathbb{N}$. To the contrary, the set $\mathbb{N}_\infty$ (Definition 6.2.3) has a countably infinite number of elements $\aleph_\mathcal{X} - n$ and a similar number of $\aleph_\mathcal{X} + n$ for each of an uncountably infinite number of $\mathcal{X}$. It is guaranteed that $n \in \mathbb{N}_\infty$ will provide a sufficient labeling scheme for $\mathbb{F}(n)$.

Proposition 7.2.11 For every $\mathbb{F}_0(n)$ or $\mathbb{F}_\aleph(n)$, the respective subset of $\mathbb{R}_0^\cup$ or $\mathbb{R}_\aleph^\cup$ whose elements are less than any x in $\mathbb{F}_0(n)$ or $\mathbb{F}_\aleph(n)$ has a supremum and the subset of $\mathbb{R}_0^\cup$ or $\mathbb{R}_\aleph^\cup$ whose elements are greater has an infimum.

Justification. We will neglect the subscripts 0 and $\aleph$ in this proof. This proposition regards the extrema of a set of sets so those extrema will be sets themselves ordered by the big parts of the nested elements $x \in \mathbb{R}^\mathcal{X} \subset \mathbb{R}^\cup$. Call $\mathbb{R}_-^\cup$ the set of all $\mathbb{R}^\mathcal{X}$ whose elements are less than any $x \in \mathbb{F}(n)$ and call the greater set $\mathbb{R}_+^\cup$. By Definition 7.2.9, $\mathbb{F}(n)$ is a connected interval and every two $\mathbb{F}(j), \mathbb{F}(k)$ are disconnected whenever $j \neq k$. Furthermore, Corollary 7.1.5 proves that every two $\mathbb{R}^\mathcal{X} \neq \mathbb{R}^\mathcal{Y}$ are disconnected. Since $[0, \infty]$ is a connected union of $\mathbb{F}(n)$ and $\mathbb{R}^\cup$, with the former being closed intervals and the latter being open, it follows that the structure of $\mathbb{R}^+$ is an ordered union

$$\mathbb{R}^+ = ...\,\mathbb{F}(n) \cup \mathbb{R}^\mathcal{X} \cup \mathbb{F}(n+1) \cup \mathbb{R}^\mathcal{Y} \cup \mathbb{F}(n+2)\,... \quad .$$

This contradicts Theorem 7.1.2, however. If there was an $\mathbb{R}^\mathcal{Z}$ between $\mathbb{R}^\mathcal{X}$ and $\mathbb{R}^\mathcal{Y}$, then it would necessarily be $\mathbb{R}^\mathcal{Z} \subset \mathbb{F}(n+1)$ contradicting the definition of $\mathbb{F}$. The connected property of $\mathbb{R}$ requires, therefore, that we introduce an alternative labeling scheme before continuing.

Definition 7.2.12 For $n \in \mathbb{N}_\infty$, the connected elements of $\mathbb{R}_0^\cup$ are labeled $\mathbb{R}_0(n)$ and the connected elements of $\mathbb{R}_\aleph^\cup$ are labeled $\mathbb{R}_\aleph(n)$. The labeling convention in either variant is such that

$$\forall x \in \mathbb{R}(n) \quad \forall y \in \mathbb{R}(m) \quad \text{s.t.} \quad n > m \quad \implies \quad x > y \quad .$$

It follows that $\mathbb{R}_0^0 = \mathbb{R}_0(1)$ and $\mathbb{R}_\aleph^0 = \mathbb{R}_\aleph(1)$. We say that $\mathbb{R}_0(n)$ is the natural neighborhood of $\aleph(n)$ and $\mathbb{R}_\aleph(n)$ is its whole neighborhood. Specifically, $\aleph(1) = \aleph_0 = 0$.

Continuing with the justification of Proposition 7.2.11, we may now infer that $[0, \infty]$ is constructed from an ordered union of the form

$$[0, \infty] = \mathbb{R}(1) \cup \mathbb{F}(1)\,...\,\mathbb{R}(n) \cup \mathbb{F}(n) \cup \mathbb{R}(n+1) \cup \mathbb{F}(n+1)\,... \quad .$$

Since the connectedness of $\mathbb{R}$ requires the sequential alternation of the disconnected $\mathbb{R}(n)$ and $\mathbb{F}(n)$ in the total ordered union, it follows that $\mathbb{R}(k)$ is the supremum of $\mathbb{R}^{\cup}_{-}$ whose elements are less than any $x \in \mathbb{F}(k)$, and $\mathbb{R}(k+1)$ is the infimum of $\mathbb{R}^{\cup}_{+}$ whose elements are greater than any $x \in \mathbb{F}(k)$. This concludes the justification of Proposition 7.2.11.

Remark 7.2.13 In Example 6.2.15, we considered the limits

$$\lim_{b\to\infty} \aleph_{\mathcal{X}} - b = -\infty \quad , \quad \text{and} \quad \lim_{b\to\widehat{\infty}} \aleph_{\mathcal{X}} - b = \aleph_{\mathcal{X}} - \widehat{\infty} = -\aleph_{(1-\mathcal{X})} \quad ,$$

as two desirable modes of limit behavior. Now the $\mathbb{F}(n)$ notation suggests a third desirable behavior such that

$$\lim_{b\to\aleph(2)} \aleph(n) + b = \aleph(n+1) \quad .$$

It may or may not be possible to accommodate this limiting mode. It might be that any sequence which does not converge its own local neighborhood of fractional distance must diverge all the way to infinity in one variety or another. Indeed, the property $\frac{d}{dx}\aleph_x = \widehat{\infty}$ (Theorem 6.2.2) suggests in some sense that once a sequence fails to converge in its local $\aleph_{\mathcal{X}}$-neighborhood, it has to keep diverging to some maximal value.

The likely issue with such a limiting mode as $b \to \aleph(2)$, something which may even amount to a flaw in the justification of Proposition 7.2.11, is that $\aleph(2) = \aleph_{\mathcal{X}_{\min}}$ is such that $\mathcal{X}_{\min}$ is the smallest positive real number. It is generally understood that no such number exists. We have developed a requirement for such a number in the course of supporting Proposition 7.2.11 but the lack of a smallest positive real number is so well-established that we might suppose no such number exists. Contrary to our conjuring of $\aleph_{\mathcal{X}}$ by requirement, there exists a large body of demonstrations that no such $\mathcal{X}_{\min}$ can exist while $\aleph_{\mathcal{X}}$ has only been supposed not to exist. If such a number as $\mathcal{X}_{\min}$ can be derived from the ordered union given in Proposition 7.2.11, then that would be very exciting. However, there are many problems associated with such a line of reasoning. We will present a few of them in Section 7.3 and then we will not use the (n) notations in a formal way moving forward.

Definition 7.2.14 This definition regards both $0, \aleph$ subscript variants of the relevant objects. To avoid the problematic (n) notation, label each connected element of $\mathbb{F}$ as $\mathbb{F}^{\mathcal{X}}$. For every $\mathbb{R}^{\mathcal{X}}$, there exists a unique $\mathbb{F}^{\mathcal{X}}$ such that

$$x \in \mathbb{R}^{\mathcal{X}} \quad , \quad z \in \mathbb{F}^{\mathcal{X}} \quad \Longrightarrow \quad x < z \quad ,$$

and

$$y \in \mathbb{R}^{\mathcal{Y}} \quad , \quad z \in \mathbb{F}^{\mathcal{X}} \quad , \quad \mathcal{Y} > \mathcal{X} \quad \Longrightarrow \quad y > z \quad .$$

In other words, there is a closed interval $\mathbb{F}^{\mathcal{X}}$ right-adjacent to every $\mathbb{R}^{\mathcal{X}}$ whenever $0 \leq \mathcal{X} < 1$. With this definition, we move the elements of $\mathbb{F}$ into the non-problematic superscript $\mathcal{X}$ labeling scheme as opposed to moving the $\mathbb{R}^{\mathcal{X}}$ into the (n) scheme as in Definition 7.2.12.

§7.3 Paradoxes Related to Infinitesimals

In this section, we demonstrate a few paradoxes, or contradictions, invoked by the $\mathbb{R}_0(n)$ enumeration scheme and its corollary concept of a least positive real number. We solve some of the paradoxes in this section with the superscript $\mathcal{X}$ label (Definition 7.2.14) and the other paradoxes are resolved in Section 7.5.

Definition 7.3.1 $\mathcal{F}(n) \in \mathbb{R}$ is the unique real number in the center of $\mathbb{F}_0(n)$ and $\mathbb{F}_\aleph(n) \subseteq \mathbb{F}_0(n)$ in the sense that for every $\mathcal{F}(n) + b \in \mathbb{F}(n)$ there exists a $\mathcal{F}(n) - b \in \mathbb{F}(n)$ (in either variant of $\mathbb{F}$.) In the alternative labeling scheme, $\mathcal{F}_\mathcal{X}$ is the number in the center of $\mathbb{F}_0^\mathcal{X}$ and $\mathbb{F}_\aleph^\mathcal{X}$. In either label, the number has the property

$$\text{Big}(\mathcal{F}_\mathcal{X}) = \mathcal{F}_\mathcal{X} \quad , \quad \text{and} \qquad \text{Lit}(\mathcal{F}_\mathcal{X}) = 0 \quad .$$

Theorem 7.3.2 *The number $\mathcal{F}(1) = \mathcal{F}_0$ has infinitesimal fractional magnitude with respect to* **AB**.

Proof. We will use Robinson's standard non-standard definition of a hyperreal infinitesimal [14,15]. A number ε is a positive infinitesimal number if and only if

$$\forall x \in \mathbb{R}^+ \quad \exists \varepsilon \notin \mathbb{R} \quad \text{s.t.} \quad 0 < \varepsilon < x \quad .$$

By construction, $\mathcal{F}(1)$ is the number in the center of the gap between $\mathbb{R}_\aleph(1) = \mathbb{R}_\aleph^0$ and $\mathbb{R}_\aleph(2) \in \{\mathbb{R}_\aleph^\mathcal{X}\}$. Since $\mathcal{F}(1)$ is not in $\mathbb{R}_\aleph(1) = \mathbb{R}_\aleph^0$, it cannot have zero fractional magnitude; $\mathbb{R}_\aleph^0$ is the set of all numbers having zero fractional magnitude along **AB**. If it had non-zero real fractional magnitude, then it would be $\mathcal{F}(1) \in \{\mathbb{R}_0^\mathcal{X}\}$, an obvious contradiction because $\mathcal{F}(1)$ has less fractional magnitude than any nested element in that set of sets. If we denote the fractional magnitude of $\mathcal{F}(1)$ with the symbol ε, the properties of this magnitude are exactly those given above in the definition of an infinitesimal. The theorem is proven. ∎

Definition 7.3.3 A number is said to be a measurable number if it can exist within the algebraic representation of some $X \in \mathbf{AB}$. If a number is not measurable, then it is immeasurable.

Theorem 7.3.4 *Every $x \in \mathbb{F}$ is an immeasurable number.*

Proof. The FDFs are bijective between their domain AB and range $[0,1] \subset \mathbb{R}$. The range is a real interval containing no numbers with infinitesimal parts so this tells us that $\mathcal{F}(n)$ is not in the algebraic representation of any geometric $X \in \mathbf{AB}$. In the $\mathcal{X}$ notation, the numbers in each $\mathbb{F}_\mathcal{X}$ have infinitesimally more fractional distance along **AB** than the numbers in each $\mathbb{R}^\mathcal{X}$. ∎

Remark 7.3.5 We have granted that every geometric point X has an algebraic representation (Axiom 2.3.11) but we have not required the opposite. Therefore, there is no problem with an infinitesimal fractional magnitude for $\mathcal{F}(1)$ because there is no corresponding point $X \in \mathbf{AB}$ that is required to have infinitesimal fractional distance. Although $\mathcal{F}(1) = \mathcal{F}_0$ has infinitesimal fractional magnitude, the number itself is very large. It is greater than any natural number.

Paradox 7.3.6 Every $\mathcal{F}(n)$ has the property

$$\mathcal{F}(n) = \frac{\aleph(n) + \aleph(n+1)}{2} \quad . \tag{7.1}$$

Every $\mathbb{R}_0^{\mathcal{X}}$ can be obtained by a translation operation on another element of $\{\mathbb{R}_0^{\mathcal{X}}\}$. Doing set-wise arithmetic, we may write, for instance

$$\hat{T}(\aleph_\delta)\mathbb{R}_0^{(\mathcal{X}-\delta)} = \aleph_\delta + \mathbb{R}_0^{(\mathcal{X}-\delta)} = \mathbb{R}_0^{\mathcal{X}} \quad .$$

Letting $AB \equiv [\aleph(n), \aleph(n+1)]$ for some $n \geq 2$, define

$$S_n = \mathbb{R}_0(n) \cap AB \quad , \quad \text{and} \qquad S_{n+1} = \mathbb{R}_0(n+1) \cap AB \quad .$$

The translational symmetry requires

$$\text{len}\big(S_n\big) = \text{len}\big(S_{n+1}\big) \quad .$$

Since

$$AB = S_n \cup \mathbb{F}_0(n) \cup S_{n+1} \quad ,$$

and since $\mathcal{F}(k)$ is the number in the center of the closed interval $\mathbb{F}_0(k)$, it is obvious that $\mathcal{F}(n)$ is the unique number in the center of the line segment AB. In the Euclidean metric, this number is always the average of the least and greatest numbers in the algebraic representations of A and B respectively. However, if the $\mathbb{R}_0(n)$ notation is a label for $\mathbb{R}_0^{\mathcal{X}}$ where $\mathcal{X}$ is strictly a real number, then, using the original labeling scheme without (n), we find

$$\mathcal{F}(n) = \frac{\aleph_{\mathcal{X}} + \aleph_{\mathcal{Y}}}{2} = \aleph_{\left(\frac{\mathcal{X}+\mathcal{Y}}{2}\right)} \quad .$$

This number is most obviously an element of $\mathbb{R}_0^{\left(\frac{\mathcal{X}+\mathcal{Y}}{2}\right)}$. This contradicts the definition $\mathcal{F}(n) \in \mathbb{F}_0(n)$. It is paradoxical that $\aleph(n+1)$ cannot have any corresponding $\aleph_{\mathcal{X}}$.

Paradox Resolution 7.3.7 Paradox 7.3.6 does not exist in the $\mathcal{F}_{\mathcal{X}}$ notation. If we never suppose the existence of $\aleph(n+1)$, then there is no starting point in Equation (7.1) and the paradox cannot be demonstrated, *i.e.*:

$$\mathcal{F}_{\mathcal{X}} \neq \frac{\aleph_{\mathcal{X}} + \aleph_{\mathcal{Y}}}{2} \quad .$$

Paradox 7.3.8 The neighborhood of the origin contains numbers of the form $\aleph_{\mathcal{X}} + b$ for b strictly non-negative (and $\mathcal{X} = 0$) but every intermediate neighborhood allows both signs for b. It follows that

$$\text{len}\big(\mathbb{R}_0(1)\big) = \frac{1}{2}\,\text{len}\big(\mathbb{R}_0(2)\big) \quad , \quad \text{and} \qquad \mathcal{F}(1) = \frac{1}{3}\,\mathcal{F}(2) \quad .$$

Every element of $\mathbb{R}_0(2)$ has positive real fractional magnitude because $\mathbb{R}_0(2) \subset \{\mathbb{R}_0^{\mathcal{X}}\}$ but if $\mathcal{F}(1)$ has infinitesimal fractional magnitude ε, then 3ε is also less than any real number (according to Robinson's arithmetic for hyperreal numbers [14, 15].) If 3ε is infinitesimal, then that contradicts the ordering

$$x \in \mathbb{R}_0(n) \qquad \Longrightarrow \qquad x < \mathcal{F}(n) \quad .$$

Paradox Resolution 7.3.9 Paradox 7.3.8 is resolved in the $\mathcal{F}_{\mathcal{X}}$ formalism. We can uniquely associate $\mathcal{F}(1) = \mathcal{F}_0$ but there is no $\mathcal{F}_{\mathcal{X}_{\text{min}}}$ that we might associate with $\mathcal{F}(2)$.

Paradox 7.3.10 Each $\mathbb{F}_0(n) \subset \mathbb{F}_0$ is a closed, connected interval. It is required, then, that

$$\mathbb{F}_0(1) = [a, b] \quad , \quad \text{and} \qquad \mathcal{F}(1) = \frac{b-a}{2} \quad .$$

Assuming the normal arithmetic for $x \in \mathbb{F}$, it follows that

$$\sup(\mathbb{R}_0) = b - 2\mathcal{F}(1) = a \quad .$$

This is paradoxical for the reasons presented in Proposition 5.4.10: $\mathbb{R}_0$ ought not have a supremum.

Paradox 7.3.11 If $\mathcal{F}(1)$ is a real number centered in the closed interval $\mathbb{F}(1)$, then, assuming the normal arithmetic for $x \in \mathbb{F}$, we find that $2\mathcal{F}(1) = \aleph_{\mathcal{X}_{\text{min}}}$ with $\mathcal{X}_{\text{min}}$ being the least positive real number. This number does not exist. Therefore, the implied identity

$$\mathcal{F}(1) = \frac{\aleph_{\mathcal{X}_{\text{min}}}}{2} \quad ,$$

is inadmissibly paradoxical.

§7.4 Complements of Natural Neighborhoods

In this section, we take many of the facts established in the previous sections and begin to put them together to form a coherent picture of $\mathbb{F}^{\mathcal{X}}$, $\mathbb{R}_0^{\mathcal{X}}$, $\mathbb{R}_\aleph^{\mathcal{X}}$, $\mathbb{R}_C^{\mathcal{X}}$, and the rest. This is what we know so far:

- We have defined $\mathbb{R}_0^{\mathcal{X}} \cup \mathbb{R}_C^{\mathcal{X}} = \mathbb{R}_\aleph^{\mathcal{X}}$.

- We do not know whether or not $\mathbb{R}_C^{\mathcal{X}} = \varnothing$. This will be the main issue decided in the present section.
- We have not yet defined any arithmetic operations for $x \in \mathbb{F}^{\mathcal{X}} \cup \mathbb{R}_C^{\mathcal{X}}$.
- We have not yet given an algebraic construction for $x \in \mathbb{F}^{\mathcal{X}} \cup \mathbb{R}_C^{\mathcal{X}}$.

Remark 7.4.1 In this section, we will use $\mathcal{F}(1) = \mathcal{F}_0$ to refer to a real number which is an upper bound on $\mathbb{R}_0$ without assuming an attendant problematic (n) enumeration scheme.

Theorem 7.4.2 *If we assume the usual arithmetic for $\mathcal{F}_{\mathcal{X}}$, then the set $\mathbb{R}_{\aleph}^0$ lies within the left endpoint A of the line segment $AB = [0, \mathcal{F}_0]$. In other words, every element of $\mathbb{R}_{\aleph}^0$ has zero fractional magnitude even with respect to $len[0, \mathcal{F}_0] \lll len\,\mathbf{AB}$.*

Proof. Every interval has a unique number at its center. For $AB = [0, \mathcal{F}_0]$, this number is $c = \frac{1}{2}\mathcal{F}_0$, as in Figure 2. If $c \in \mathbb{R}_{\aleph}^0$, meaning that the fractional magnitude with respect to **AB** was zero, then $2c = \mathcal{F}_0$ would also have $2 \times 0 = 0$ fractional magnitude with respect to **AB**. This is contradictory because it would require $\mathcal{F}_0 \in \mathbb{R}_{\aleph}^0$. Continuing the argument, we find that for any $n \in \mathbb{N}$, the number $\frac{1}{n}\mathcal{F}_0$ must not be an element of $\mathbb{R}_{\aleph}^0$. Now assume $\frac{1}{n}\mathcal{F}_0 \in X \in AB \equiv [0, \mathcal{F}_0]$ and $X \neq A$. Since the quotient of two line segments is defined as a real number (Definition 3.1.1), and since the difference of two real numbers is always greater than some inverse natural number (Axiom 2.1.6), we may write for some $m \in \mathbb{N}$

$$\frac{AX}{AB} - \frac{AA}{AB} > \frac{1}{m} \quad .$$

This is satisfied for any $X \neq A$ so $\frac{1}{n}\mathcal{F}_0$ can be a number in the algebraic representation of any $X \neq A$. Since $\frac{1}{n}\mathcal{F}_0 \notin \mathbb{R}_{\aleph}^0$, $\mathbb{R}_{\aleph}^0$ must be a subset of, or equal to, the algebraic representation of the left endpoint A of $AB \equiv [0, \mathcal{F}_0]$.

Corollary 7.4.3 *If we assume the usual arithmetic for $\mathcal{F}_{\mathcal{X}}$, then for any $x \in \mathbb{R}_0$ such that $x \in X$, and for $X \in AB$ such that $AB \equiv [0, \mathcal{F}_0]$, we have*

$$\mathcal{D}_{AB}(AX) = 0 \quad .$$

Proof. By the property $\mathbb{R}_0^0 \subseteq \mathbb{R}_{\aleph}^0$, proof follows from Theorem 7.4.2.

Alternatively, $\mathcal{D}_{AB}$ is such that

$$\mathcal{D}_{AB}(AX) = \mathcal{D}_{AB}^{\dagger}(AX) = \frac{x}{\mathcal{F}_0} \quad .$$

The case of $x = 0$ is trivial. To prove the other cases by contradiction, suppose $z > 0$ and that

$$\frac{x}{\mathcal{F}_0} = z \quad .$$

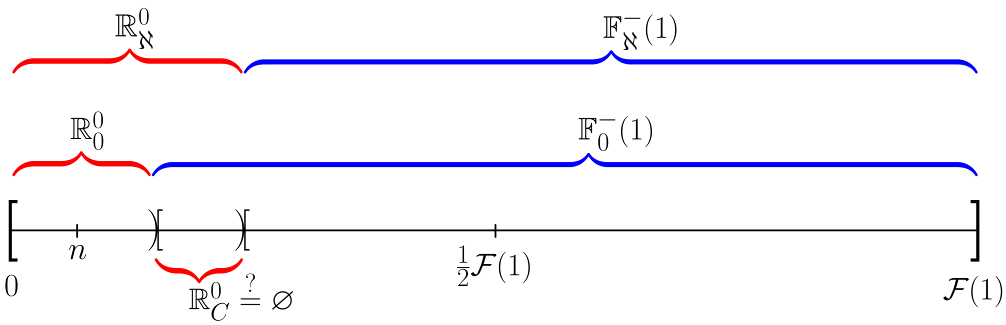

Figure 2: This figure (*not to scale!*) shows the neighborhood of the origin $\mathbb{R}^0_\aleph$, the substructure of that neighborhood, and the associated structure from the Cantor-like sets. $\mathbb{F}^-$ refers to the subset of $\mathbb{F}$ which is less than or equal to $\mathcal{F}(1) = \mathcal{F}_0$.

Since $\|x\| < \|\mathcal{F}_0\|$ and $x, \mathcal{F}_0 \in \mathbb{R}^+$, it follows that $0 < z < 1$. All such $z \in \mathbb{R}_0$ have a multiplicative inverse $z^{-1} \in \mathbb{R}_0$ so

$$\frac{x}{z\mathcal{F}_0} = 1 \qquad \Longleftrightarrow \qquad z^{-1}x = \mathcal{F}_0 \quad .$$

This is a contradiction because $z^{-1}x \in \mathbb{R}_0$ but $\mathcal{F}_0$ is greater than any element of $\mathbb{R}_0$.

Remark 7.4.4 Suppose we define $\digamma_\mathcal{X} = \mathcal{X} \cdot \mathcal{F}_0$ so that it mirrors the structure of $\aleph_\mathcal{X} = \mathcal{X} \cdot \widehat{\infty}$. Since $\mathbb{R}^0_\aleph$ has zero fractional distance even along $AB \equiv [0, \mathcal{F}_0]$, we could define a set of whole neighborhoods along AB

$$\mathbb{R}^\mathcal{X}_\mathcal{F} = \left\{ \digamma_\mathcal{X} + b \;\middle|\; b \in \mathbb{R}^0_\aleph \right\} \quad ,$$

exactly dual to the elements of $\{\mathbb{R}^\mathcal{X}_\aleph\}$ spaced along $\mathbf{AB} \equiv [0, \infty]$. By subtracting every $\mathbb{R}^\mathcal{X}_\mathcal{F}$ from the interval $[0, \mathcal{F}_0]$ we would create another Cantor-like set. Following the prescription given in Section 7.2, we would invoke the connectedness of the interval to label the disconnected elements of the new Cantor-like set, and we would label the numbers in the centers of each of those disconnected intervals. Call those number $\mathcal{G}(n)$ labeled with $n \in \mathbb{N}_\infty$ so that they are dual to the $\mathcal{F}(n)$ in the duality transformation $[0, \infty] \to [0, \mathcal{F}_0]$, and so that they have a non-problematic labeling scheme $\mathcal{G}_\mathcal{X}$ with $\mathcal{X}$ measuring fractional distance along $AB \equiv [0, \mathcal{F}_0] \not\equiv \mathbf{AB}$. By replicating the present course of analysis, we could show that no element of $\mathbb{R}^0_\aleph$ has non-zero fractional magnitude even with respect to $\text{len}[0, \mathcal{G}_0] \lll \text{len}[0, \mathcal{F}_0] \lll \text{len}(\mathbf{AB})$. We could do this forever—defining more and more, tinier and tinier Cantor-like sets—and $\mathbb{R}^0_\aleph$ would never leave the left endpoint A of the line segment whose algebraic representation is $[0, \Gamma_0]$ with Γ_0 being the number in the center of the leftmost connected element of the umpteenth Cantor-like set.

To accommodate the interpretation of the positive branch of $\mathbb{R}$ as a Euclidean line segment, we were forced to introduce numbers in the form $\aleph_\mathcal{X}$. As a consequence, we were forced to introduce numbers of the form $\mathcal{F}(n) = \mathcal{F}_\mathcal{X}$ to describe the numbers in the Cantor-like sets whose elements are $\mathbb{F}_\mathcal{X}$. If we tried to define $\mathbb{F}_\mathcal{X}$ as a neighborhood of the form

$$\mathbb{F}_\mathcal{X} \stackrel{?}{=} \left\{ \mathcal{F}_\mathcal{X} + b \;\middle|\; |b| \in \mathbb{R}^0_\aleph \right\} \quad ,$$

then we would immediately encounter a problem. This set is clearly open while we have already proven that it must be closed (Theorem 7.2.7.) Therefore, we are left with a mystery set $\mathbb{F}_0$ whose elements are not easily decided. Since this is the second such set we have, $\mathbb{R}^\mathcal{X}_C$ being the first, we ought to combine them into a single mystery set. We will conjecture $\mathbb{R}^\mathcal{X}_\aleph \setminus \mathbb{R}^\mathcal{X}_0 = \varnothing$ which will be tantamount to axiomatizing it in advance of the reliance on the conjecture which follows. With this conjecture (Conjecture 7.4.5), every number which is not in a natural neighborhood yet is greater than the all the numbers in some natural neighborhood, and less than all the numbers in another, must be in some $\mathbb{F}^\mathcal{X}$ between them.

We have not proven that $\mathbb{R}^\mathcal{X}_\aleph \setminus \mathbb{R}^\mathcal{X}_0 = \varnothing$ but neither have we proven the existence of such numbers ($\mathbb{R}^\mathcal{X}_\aleph \setminus \mathbb{R}^\mathcal{X}_0 \neq \varnothing$) as we have with $\aleph_\mathcal{X}$ and $\mathcal{F}_\mathcal{X}$. We required with Axiom 4.2.6 that every $x \in \mathbb{R}$ may be constructed algebraically as a Cartesian product of Cauchy equivalence classes of rational numbers, or as a partition of all such products. So far, we have only found such constructions for those numbers in the natural neighborhoods. To avoid needless complication, therefore, we will conjecture that $\mathbb{R}^\mathcal{X}_C$ is the empty set and that, consequently, $\mathbb{F}_0 = \mathbb{F}_\aleph$. Then we will have all of the $\mathbb{R}^\mathcal{X}_0 = \mathbb{R}^\mathcal{X}_\aleph = \mathbb{R}^\mathcal{X}$ neighborhoods cleanly defined as ordered pairs of subsets of $C_\mathbb{Q}$, and we will move everything else into the Cantor-like set $\mathbb{F}$. By the following conjecture, we will have given algebraic constructions and arithmetic axioms for every number in $\mathbb{R}^\cup_0 = \mathbb{R}^\cup_\aleph$. Everything which remains to be completed is transferred by Conjecture 7.4.5 into $\mathbb{F}$.

Conjecture 7.4.5 Every number having zero fractional magnitude with respect to **AB** is an element of $\mathbb{R}_0$. Most generally,

$$\mathbb{R}^\mathcal{X}_\aleph = \mathbb{R}^\mathcal{X}_0 \quad , \quad \text{and} \qquad \mathbb{R}^\mathcal{X}_C = \varnothing \quad .$$

Remark 7.4.6 One would also want to conjecture the countercase to Conjecture 7.4.5 and examine the requirements for establishing naturally numbered tiers of increasingly large numbers, larger than any natural number but still having zero fractional distance with respect to **AB**. However, we will take the opposite tack here. Now that we have conjectured that the whole and natural neighborhoods are the same, we will drop the 0 and $\aleph$ subscripts from their respective objects.

§7.5 Immeasurable Numbers and The Least Upper Bound Problem

Axiom 4.2.6 gave $x \in \mathbb{R}$ as a Cartesian product $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ of Cauchy equivalence classes of rational numbers, or as a partition of all such products. In this section, we will invoke the partition clause to define the immeasurable $x \in \mathbb{F}$ as Dedekind partitions of $C_{\mathbb{Q}}^{\mathbf{AB}}$. We have defined the arithmetic of the equivalence classes themselves in Section 5.5 but we have not proven that partitions obey the arithmetic axioms in the way that we have have for the direct equivalence classes $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} = C_{\mathbb{Q}} \times C_{\mathbb{Q}}$. In this section, we will prove that the immeasurables do not, and cannot, obey the arithmetic axioms. This will be due mainly to the least upper bound problem (Proposition 5.4.10) which we will solve and avoid. We will prove that every $x \in \mathbb{F}$ is some $\mathcal{F}_{\mathcal{X}}$ or that, equivalently, $\mathbb{F} = \{\mathcal{F}_{\mathcal{X}}\}$ meaning that there are no connected subsets of $\mathbb{F}$ or, in other words, that $\mathbb{F}$ is a discrete set of numbers. This will require that each successive pair of $\mathbb{R}^{\mathcal{X}}$ share an extremum; the supreumum of one is the infimum of the next. Then we will suggest that the $x \in \mathbb{F}$ can be used as a set of analogue natural numbers for the formal construction of a chart conformally related to the original Euclidean chart by a scale factor $\mathcal{F}_0$ where $\mathcal{F}_0 = \mathcal{F}(1)$ is the least immeasurable number. Motivated by the regular spacing of the $\mathcal{F}_{\mathcal{X}} = \mathcal{F}(n)$, we would define (hypothetically) a transfinite version of $\mathbb{N}$, call it $\mathbb{N}_{\mathbb{T}}$, whose unit increment is such that natural numbers $n \in \mathbb{N}$ have vanishing fractional distance with respect to $1 \in \mathbb{N}_{\mathbb{T}}$. Then we would define zero as the least number in the algebraic representation of the left endpoint of some AB to infer an analogue $\mathbb{Q}_{\mathbb{T}}$ supporting the construction of $\mathbb{T}$-labeled analogues of $C_{\mathbb{Q}}$ and $C_{\mathbb{Q}}^{\mathbf{AB}}$. In this way, the formal algebraic construction of a transfinite number system follows as direct consequence of $\mathbb{R}$ without any further extraneous input beyond the initial supposition that $\mathbb{N}$ does exist. We will conclude this section showing that the immeasurables conform to the requirements of the Archimedes property of real numbers.

Definition 7.5.1 A Dedekind cut is a partition of the rationals $\mathbb{Q}$ into two sets L and R such that every real number is equal to some partition $x = (L, R)$ with the following properties.

- $L \neq \mathbb{Q}$ is non-empty.
- If (i) $x, y \in \mathbb{Q}$, (ii) $x < y$, and (iii) $y \in L$, then $x \in L$.
- If $x \in L$, then there exists $y \in L$ such that $y > x$.

Definition 7.5.2 An extended Dedekind cut is a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$ into two sets L and R such that every real number is equal to some partition $x = (L, R)$ with the following properties.

- $L \neq C_{\mathbb{Q}}^{\mathbf{AB}}$ is non-empty.
- If (i) $x, y \in C_{\mathbb{Q}}^{\mathbf{AB}}$, (ii) $x < y$, and (iii) $y \in L$, then $x \in L$.

- If $x \in L$, then there exists $y \in L$ such that $y > x$.

Theorem 7.5.3 *The number $\mathcal{F}_0 \in \mathbb{R}$ is an extended Dedekind partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$.*

Proof. Let

$$L_0 = \Big\{[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\Big|\; \mathrm{Big}(x) = 0\Big\} \;\;, \quad \text{and} \qquad R_0 = \Big\{[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\Big|\; \mathrm{Big}(x) > 0\Big\} \;\;,$$

Dedekind himself wrote the following [3].

> "In every case in which a cut (A_1, A_2) is given that is not produced by a rational number, we create a new number, an irrational number a, which we consider to be completely defined by this cut; we will say that the number corresponds to this cut or that it produces the cut.'

In that vein, the cut (L_0, R_0) is not produced by a measurable number $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ so we create the new number $\mathcal{F}_0$: an immeasurable number. We say that the cut "produces" this number or vice versa. Therefore, $\mathcal{F}_0 = (L_0, R_0)$ and the theorem is proven.

Definition 7.5.4 The extended Dedekind form of $\mathcal{F}_{\mathcal{X}} = (L_{\mathcal{X}}, R_{\mathcal{X}}) \in \mathbb{R}$ is such that

$$L_{\mathcal{X}} = \Big\{[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\Big|\; \mathrm{Big}(x) \leq \aleph_{\mathcal{X}}\Big\}$$
$$R_{\mathcal{X}} = \Big\{[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\Big|\; \mathrm{Big}(x) > \aleph_{\mathcal{X}}\Big\} \;\; .$$

Main Theorem 7.5.5 *$\mathcal{F}_{\mathcal{X}}$ is the only number in $\mathbb{F}^{\mathcal{X}} \subset \mathbb{R}$. In other words, $\mathbb{F}^{\mathcal{X}} = \mathcal{F}_{\mathcal{X}}$ or, equivalently, $\mathbb{F} = \{\mathcal{F}_{\mathcal{X}}\}$.*

Proof. It will suffice to prove this theorem if we show that $\mathbb{F}^{\mathcal{X}}$ is a one-point set. Definition 7.2.14 gives

$$x \in \mathbb{R}^{\mathcal{X}} \;\;, \quad z \in \mathbb{F}^{\mathcal{X}} \qquad \Longrightarrow \qquad x < z \;\;,$$

and

$$y \in \mathbb{R}^{\mathcal{Y}} \;\;, \quad z \in \mathbb{F}^{\mathcal{X}} \;\;, \quad \mathcal{Y} > \mathcal{X} \qquad \Longrightarrow \qquad y > z \;\; .$$

This definition establishes that $x \in \mathbb{F}^{\mathcal{X}}$ is an upper bound on $\mathbb{R}^{\mathcal{X}}$ and a lower bound on $\mathbb{R}^{\mathcal{Y}}$ for any $\mathcal{Y} > \mathcal{X}$. If $x \in \mathbb{F}^{\mathcal{X}}$ is simultaneously (i) the least upper bound of $\mathbb{R}^{\mathcal{X}}$, and (ii) the greatest lower bound of some $\mathbb{R}^{\mathcal{Y}}$, then $x = \mathcal{F}_{\mathcal{X}}$ is the unique $x \in \mathbb{F}^{\mathcal{X}}$ and the proof will be completed. For proof by contradiction, assume $u \in \mathbb{R}$ is an upper bound on $\mathbb{R}^{\mathcal{X}}$ with the property

$$u < \mathcal{F}_{\mathcal{X}} \;\; .$$

By Axiom 4.2.6, u must be a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$ or an equivalence class therein. We will divide the proof, therefore, into two parts.

• (Equivalence class) If u was $[u] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$, then $u \in \mathbb{R}^{\mathcal{Z}} \subset \mathbb{R}^{\cup}$. If $\mathcal{Z} > \mathcal{X}$, then $u > \mathcal{F}_{\mathcal{X}}$, a contradiction. If $\mathcal{X} \geq \mathcal{Z}$, then u is not an upper bound on $\mathbb{R}^{\mathcal{X}}$, another contradiction. Now it is proven that $u \neq [u] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$.

• (Partition) The partition definition is

$$u = (L_u, R_u) \quad .$$

If $u < \mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{\mathcal{X}} = (L, R)$, then there exists $\Sigma \in L$ such that

$$L_u = L \setminus \Sigma \quad , \quad \text{and} \qquad R_u = R \cup \Sigma \quad . \tag{7.2}$$

From Definition 7.5.4, we have

$$L = \Big\{ [x] \subset C_{\mathbb{Q}}^{\mathbf{AB}} \;\Big|\; \text{Big}(x) \leq \aleph_{\mathcal{X}} \Big\} \quad .$$

We have already ruled out $[u] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$. There is no $\Sigma \subset L$ which can satisfy Equation (7.2). We contradict the supposition that such a Σ does exist. Since there is no upper bound on $\mathbb{R}^{\mathcal{X}}$ less than $\mathcal{F}_{\mathcal{X}}$, $\mathcal{F}_{\mathcal{X}}$ must be the least upper bound of $\mathbb{R}^{\mathcal{X}}$.

A similar demonstration proves that $\mathcal{F}_{\mathcal{X}}$ must be the greatest lower bound of some $\mathbb{R}^{\mathcal{Y}}$ with $\mathcal{Y} > \mathcal{X}$. It follows that $\mathbb{F}^{\mathcal{X}}$ is a one-point set. The theorem is proven.

Remark 7.5.6 The least upper bound problem rears its comely head. With Main Theorem 7.5.5, we have given $\mathcal{F}_0 = \sup(\mathbb{R}_0)$ but we have already made a strong case that no such supremum can exist (Proposition 5.4.10.) In the remainder of this section, we will conclude the development of the fractional distance approach to $\mathbb{R}$ such that the reasoning behind the least upper bound problem is carefully sidestepped.

Before we continue, we will outline exactly what it means "to conclude the development." With Main Theorem 7.5.5, we have now given a construction by Cauchy equivalence classes for every $x \in \mathbb{R}$. Every measurable $x \in \mathbb{R}^{\cup}$ is directly a subset of $C_{\mathbb{Q}}^{\mathbf{AB}}$. The arithmetic of such numbers is given in Section 5. Every immeasurable $x \in \mathbb{F}$ is now formally constructed as an extended Dedekind partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$. Since $\mathbb{R} = \mathbb{R}^{\cup} \cup \mathbb{F}$, all real numbers now have a direct algebraic construction: we assume $\mathbb{N}$, define 0, construct $\mathbb{Q}$, $C_{\mathbb{Q}}$, and $C_{\mathbb{Q}}^{\mathbf{AB}}$ hence, and then we take $x \in \mathbb{R}$ as the elements and partitions of $C_{\mathbb{Q}}^{\mathbf{AB}}$. In Section 5.5, however, it was only proven that the equivalence classes themselves obey the arithmetic axioms. We cannot simply throw $\mathcal{F}_{\mathcal{X}}$ in there—not in any rigorous fashion—because there is not a Cauchy equivalence class $[\mathcal{F}_{\mathcal{X}}] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ for any $[\mathcal{X}] \subset C_{\mathbb{Q}}$. Even if we forced the arithmetic axioms onto $\mathcal{F}_{\mathcal{X}}$ with more axioms, we would immediately hit the least upper bound problem developed in Proposition 5.4.10. Now we will solve the least upper bound problem and begin moving toward the Archimedes property of $\mathbb{F} \subset \mathbb{R}$.

Definition 7.5.7 If a real number is $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$, then it is called an arithmatic number (pronounced arith·matic.) All measurable numbers $\mathbb{R}^{\cup} = C_{\mathbb{Q}}^{\mathbf{AB}}$ are arithmatic. If a real number is a partition of $C_{\mathbb{Q}}^{\mathbf{AB}}$ not given by any subset therein, then it is called a non-arithmatic number. All immeasurable numbers are non-arithmatic.

Axiom 7.5.8 Arithmetic operations are not defined among arithmatic and non-arithmatic numbers.

Definition 7.5.9 If $\mathbb{R}$ has the least upper bound property, then every non-empty subset $S \subset \mathbb{R}$ that has an upper bound must have a least upper bound $u \in \mathbb{R}$ such that $u = \sup(S)$.

Proposition 7.5.10 (Restatement of the least upper bound problem.) $\mathbb{R}$ does not have the least upper bound property because $\mathbb{R}_0$ cannot have a supremum if arithmetic is defined for $x \in \mathbb{F}$ in the usual way. If $\mathcal{F}_0 = \sup(\mathbb{R}_0)$, and if $\mathbb{F}$ were to obey the arithmetic axioms in compositions with $x \in \mathbb{R}^{\cup}$, then $\mathcal{F}_0 - 1$ would be an element of $\mathbb{R}_0$. By the closure of $\mathbb{R}_0$, $\mathcal{F}_0 - 1 + 2$ would be another element of $\mathbb{R}_0$. This contradicts the identity $\mathcal{F}_0 = \sup(\mathbb{R}_0)$ because $\mathcal{F}_0 < \mathcal{F}_0 + 1$.

Refutation. Let x be an arithmatic real number. Axiom 7.5.8 is such that

$$\mathcal{F}_0 = \sup(\mathbb{R}_0) \quad \Longrightarrow \quad \sup(\mathbb{R}_0) \pm x = \text{undefined} \quad .$$

The arithmetic axioms cannot be used to demonstrate the condition in the justification of this proposition. $\mathbb{R}_0$ ***most certainly can have a supremum***. The supremum of each open set of numbers $\mathbb{R}^{\mathcal{X}}$ that are $100 \times \mathcal{X}\%$ of the way down the real number line is $\mathcal{F}_{\mathcal{X}}$: a non-arithmatic number.

Theorem 7.5.11 $\mathbb{R}$ *has the least upper bound property which is also called the Dedekind property or Dedekind completeness.*

Proof. The Dedekind property requires that if (i) L and R are two non-empty subsets of $\mathbb{R}$ such that $\mathbb{R} = L \cup R$, meaning that (L, R) is a partition of $\mathbb{R}$, and (ii)

$$x \in L \quad , \quad y \in R \quad \Longrightarrow \quad x < y \quad ,$$

then either L has a greatest member or R has a least member. This property is implicit in the connectedness of the algebraic interval. With Main Theorem

7.5.5, we have established the connectedness of the successive disconnected intervals in $\mathbb{R}^{\cup}$. They are connected by the elements of $\mathbb{F}$. All subsets of $\mathbb{R}$ with an upper bound also have a least upper bound because $\{\mathcal{F}_{\mathcal{X}}\}$ guarantees the simple connectedness. The connectedness proves the theorem.

Remark 7.5.12 The non-arithmatic immeasurable numbers inherit their ordering with respect to the $\leq$ relation from the total ordering of $\mathbb{R}$. The supremum of one measurable neighborhood is the infimum of the next. The maximal neighborhood $\mathbb{R}^1$ does not have a supremum but it is exempted from the Dedekind property because it does not have a real upper bound at all. The upper bound of the maximal neighborhood of infinity diverges. Since no upper bound $u \in \mathbb{R}$ exists for $\mathbb{R}^1$ at all, a least upper bound on $\mathbb{R}^1$ cannot exist.

Definition 7.5.13 A set S is totally ordered if it obeys the following order axioms.

- (O1a) Elements of S have trichotomy: If $x, y \in S$, then one and only one of the following is true: (i) $x < y$, (ii) $x = y$, or (iii) $x > y$.
- (O2a) The $<$ relation is transitive: If $x, y, z \in S$, then $x < y$ and $y < z$ together imply $x < z$.
- (O3a) If $x, y, z \in S$, then $x < y$ implies $x + z < y + z$ or at least one sum is undefined.
- (O4a) If $x, y, z \in S$, and if $z > 0$, then $x < y$ implies $xz < yz$ or at least one product is undefined.

Remark 7.5.14 Axioms (O1a)-(O4a) are almost exactly the (O1)-(O4) ordering axioms of a complete ordered number field (Axiom 5.4.7.) We have changed the two axioms involving the arithmetic operations $+$ and $\times$. The changes make allowances for the immeasurable $x \in \mathbb{F}$ which are truncated from existence with a requisite field-related axiom that every real number is less than some natural number.

Theorem 7.5.15 $\mathbb{R}$ *is a totally ordered set.*

Proof. We will prove each of (O1a) to (O4a).

• (O1a) Trichotomy is trivially inherent to the order established for $\mathbb{R}^{\cup}$ (Axiom 5.2.14). Trichotomy is fully satisfied in $\mathbb{R} = \mathbb{R}^{\cup} \cup \mathbb{F}$ by the result that $\mathcal{F}_{\mathcal{X}} = \sup(\mathbb{R}^{\mathcal{X}})$.

• (O2a) Transitivity is satisfied by Axiom 5.2.14 and the corollary results for the extrema $\{\mathcal{F}_{\mathcal{X}}\}$.

• (O3a) Since we have restricted this part of Definition 7.5.13 to the arithmatic numbers, the arithmetic axioms give compliance as stated.

• (O4a) Satisfaction follows in the manner of (O3a).

The ordering relation $\leq$ for $\mathbb{R}$ is such that $\mathbb{R}$ is totally ordered.

Remark 7.5.16 In Definition 7.5.13, we have modified slightly the usual definition of total order (Axiom 5.4.7) so that (O3a) and (O4a) distinguish arithmatic and non-arithmatic numbers. We will justify this exception to the usual definition of total order as follows.

Since we are not using a number field approach to $\mathbb{R}$, we need not state the definition of total order in the exact form of the axioms of a totally ordered complete number field having unified laws of arithmetic. The lack of arithmetic definitions for immeasurable numbers doesn't affect their well-ordering with respect to the measurable ones. The lack of defined operations has no bearing on the concept of the ordering of the set. Indeed, regarding the geometric notions of addition and multiplication which we have referred to throughout this analysis of fractional distance, there should not exist geometrically identical arithmetic operations for numbers with geometrically immeasurable fractional distance. Shared geometry-based arithmetic operations would necessarily imply a shared character of measurablity or immeasurablity, but not both characteristics shared simultaneously. The requirement for this shared character is the root of the discrepancy in the least upper bound problem. In our development of the least upper bound problem (Proposition 5.4.10), it was implicitly assumed that $\sup(\mathbb{R}_0)$ must be an arithmatic number in the way that ***all real numbers were supposed to be algebraic until the connectedness of the interval demanded non-algebraic numbers to fill in the gaps***. A number is said to be canonically algebraic if it is the root of a certain polynomial. This is not what is presently meant using the adjective "algebraic" to describe the immeasurables where the word refers to the lack of a geometric picture of fractional distance for $x \in \mathbb{F}$. The geometric picture of $x \in \mathbb{F}$ comes from the algebraic ordering with the $\leq$ relation being defined over all of $\mathbb{R}$. It is clear that the $\mathcal{F}_{\mathcal{X}}$ are geometric Euclidean magnitudes, or cuts, measured relative to the origin $\hat{0}$ of an infinitely long line. However, it is not clear how this works in the metric space picture of $\mathbb{R}$ where

$$d(0, \mathcal{F}_{\mathcal{X}}) = |\mathcal{F}_{\mathcal{X}} - 0| = \text{undefined} \quad .$$

Toward the metric space picture, it is interesting that we have shown in Main Theorem 6.1.1 that arithmetic in the neighborhood of infinity allows us to take all-important limits at infinity within the realm of standard analysis. There is no need to invoke the metric space definition of $\mathbb{R}$ to take these limits. The metric space is the canonical workaround for a supposed failure of the Cauchy limit criterion at infinity but here we take the Cauchy limit at infinity with the modernized definition of $\mathbb{R}$ (Main Theorem 6.1.1.) Why, then, should it be a problem that the metric function is undefined? In Definition 2.1.2, we defined a number line as a line equipped with a chart x and the Euclidean metric $d(x, y) = |y - x|$ but now that we have closely examined all the details, we can otherwise define a number line as a line equipped with the totally ordered fractional distance chart $\pm x \in \mathbb{R}^{\cup} \cup \mathbb{F}$ (which is just the Euclidean chart.) With this definition taken *a priori* as an axiom, it is possible to reproduce the entire fractional distance analysis without any dependence on a metric whose domain only holds the measurables $\mathbb{R}^{\cup}$. If desired, it could be assumed that the metric definition of a number line (Definition 2.1.2) is overwritten as needed for consistency. By the end of this section, however, we will have restored the metric functionality.

What we have done with the separation of the reals into arithmatic and non-arithmatic numbers mirrors the usual separation between algebraic numbers, which are the roots of non-zero polynomials with rational coefficients, and canonically non-algebraic numbers which are not the roots of any such polynomials. Canonically non-algebraic numbers are supposed to exist because they are needed to fill in the gaps in $\mathbb{Q}$ which are not allowed if $\mathbb{R}$ is to satisfy the definition of a simply connected 1D interval. Now we have gone one step further and shown that ***non-arithmatic numbers are needed to fill in the connectedness of the many neighborhoods***.

Paradox Resolution 7.5.17 (Resolution of Paradox 7.3.10.) The paradox depends on assumed usual arithmetic for non-arithmatic immeasurable numbers. The paradox is remedied by the non-arithmatic property of $\mathcal{F}_{\mathcal{X}}$.

Paradox Resolution 7.5.18 (Resolution of Paradox 7.3.11.) The paradox depends on assumed usual arithmetic for non-arithmatic immeasurable numbers. The paradox is remedied by the non-arithmatic property of $\mathcal{F}_{\mathcal{X}}$.

Remark 7.5.19 Throughout this analysis, we have referred many times to the geometric notions of addition and multiplication. If $\mathcal{F}_{\mathcal{X}}$ is an immeasurable number $x \notin \mathbb{R}^{\cup}$ such that ordinary notions of geometry cannot be applied to it universally, by what means might we axiomatize the arithmetic of non-arithmatic numbers? We have shown in Section 7.3 that the straightforward introduction of infinitesimals is not the correct way forward, and we have shown it for all the reasons that infinitesimals are usually not allowed into standard analysis. So, to pierce the reader's probable veil of likely skepticism regarding some perceived absurdity of ordered but non-arithmatic real numbers $\mathcal{F}_{\mathcal{X}}$—which are not one iota more specious a construction than the non-algebraic numbers which were brought into standard analysis for the *exact* same reason—we now point out that the $\{\mathcal{F}_{\mathcal{X}}\}$ are the only real numbers not forced into the arithmetic axioms by some identification as an element of $C_{\mathbb{Q}}^{\mathbf{AB}}$. Immeasurable real numbers $x \in \mathbb{F}$ are partitions of the big parts of $C_{\mathbb{Q}}^{\mathbf{AB}}$ only. They are not uniquely identified with any $[x] \in C_{\mathbb{Q}}^{\mathbf{AB}}$ whose corresponding Dedekind partition would specify a big part *and* a little part. Measurable numbers differ from immeasurable numbers because the Dedekind partition corresponding to any $x \in \mathbb{R}^{\cup}$ must specify a little part. Measurable and immeasurable numbers are so markedly different in their qualia that it is certainly reasonable to suppose that they obey different arithmetic axioms.

Remark 7.5.20 Now, in advance of the material presented in the remainder of this section, we will laboriously belabor the point that ***there is no preferred scale for standard analysis***. As mentioned in the preamble to this section, we will deem to take the immeasurables $\mathcal{F}(n)$ as the extended naturals on a real line whose unit of distance is equal in magnitude to the $\mathcal{F}_0 = \mathcal{F}(1)$ of the real line we have described already. We will use the label $\mathbb{R}_{\mathbb{T}}$ to distinguish the new copy of $\mathbb{R}$.

Geometric infinity ∞ is not affected in any way, ever, by any conformal rescaling parameter. From a geometric perspective, then, we should expect that rescaled $\mathbb{R}_{\mathbb{T}}$ should be $\mathbb{R}$ itself identically. For instance, consider a convention such that

$$\mathbb{R} = (-\aleph_1, \aleph_1) \quad , \quad \text{and} \qquad \mathbb{T} = (-\aleph_{\widehat{\infty}}, \aleph_{\widehat{\infty}}) \quad ,$$

where the existence of some $\mathbb{T}$ separate from $\mathbb{R}$ is implied by the notion of geometric infinity. If $\mathbb{R}$ had an endpoint at geometric infinity, then that would contradict the notion of infinite geometric extent forbidding endpoints at ∞. The limit definition of algebraic infinity

$$\left|\widehat{\infty}\right| = \lim_{x\to 0^{\pm}} \frac{1}{x} \quad ,$$

is such that the scale of the unit $1 \in \mathbb{N}$ in the numerator does not affect our ability to construct $(-\aleph_1, \aleph_1) \subseteq (-\aleph_\infty, \aleph_\infty)$. However large or small we take the scale of $1 \in \mathbb{N}$, the resultant extended real line $[-\widehat{\infty}, \widehat{\infty}]$ will always be a subset of $\mathbb{T}$. Regardless of the scale of $1 \in \mathbb{N}$, the interval $(-\widehat{\infty}, \widehat{\infty})$ is always going to be the same set $\mathbb{R}$, even if we add supplemental labeling to denote the relative scale of the different copies of $\mathbb{R}$, and even if the relative scale is on the order of $\mathcal{F}_0$ or greater. There is no scale factor by which we might stretch algebraic infinity as in $\mathbb{R} = (-\widehat{\infty}, \widehat{\infty})$ to be more than a drop in the bucket with respect to geometric infinity as in $\mathbb{T} = (-\aleph_{\widehat{\infty}}, \aleph_{\widehat{\infty}})$. For every instance of $\widehat{\infty} \in \overline{\mathbb{R}}$ such that $|\widehat{\infty}| = \infty$, there is a conformal chart containing points to the right of $\widehat{\infty}$ such that those points are to the left of the ∞ of the other chart. The interval containing those transfinite points will always be infinitely longer than the scaled instance of $\mathbb{R}$.

If we have two copies of the real line called $\mathbb{R}_4$ and $\mathbb{R}_{13}$ such that the unit of Euclidean distance in $\mathbb{R}_{13}$ is $\frac{13}{4}$ longer than the unit of distance in $\mathbb{R}_4$, then that does not affect the perfect individual compliance of either copy with the definition of $\mathbb{R}$. The same is true if the interval $[0,1]$ in one copy is as long as $[0,\mathcal{F}_0]$ in the other. In the remainder of this section, we will infer that the immeasurables behave like the naturals on a copy of $\mathbb{R}$ which is quite large relative to another copy: one scaled down by $\mathcal{F}_0$. We will define $\mathcal{F}(n)=n\in\mathbb{N}_\mathbb{T}$ and then claim that $\mathbb{N}_\mathbb{T}=\mathbb{N}$ because there is no unique scale for $\mathbb{R}$. Having presupposed the existence of natural numbers at the outset our construction by Cauchy sequences of rationals, we will find that the curious remainder set $\mathbb{F}$ left over at the end is exactly what we have put in to begin with. At the end, we will find that the abstract set of natural numbers which served as the starting point for $C_\mathbb{Q}^{\mathbf{AB}}$ were simply the $\mathcal{F}(n)$ on a smaller copy of $\mathbb{R}$ whose existence can be inferred by the self-similarity of $\mathbb{R}$ on any scale. It's tortoises all the way down.

We have demonstrated by the least upper bound problem a requirement that standard analysis must not contain any compositive Cartesian products in the forms

$$\mathbb{R}^\cup\times\mathbb{F}=\begin{cases}\{x+\mathcal{F}_\mathcal{X}\mid x=[x],\ \mathcal{F}_{[\mathcal{X}]}\in\mathbb{F},\ [x],[\mathcal{X}]\subset C_\mathbb{Q}^{\mathbf{AB}}\}\\ \{x\cdot\mathcal{F}_\mathcal{X}\mid x=[x],\ \mathcal{F}_{[\mathcal{X}]}\in\mathbb{F},\ [x],[\mathcal{X}]\subset C_\mathbb{Q}^{\mathbf{AB}}\}\\ \{x\div\mathcal{F}_\mathcal{X}\mid x=[x],\ \mathcal{F}_{[\mathcal{X}]}\in\mathbb{F},\ [x],[\mathcal{X}]\subset C_\mathbb{Q}^{\mathbf{AB}}\}\end{cases}\quad.$$

Accordingly, we have developed an axiomatic framework which forbids these compositions. The fractional distance axioms are such that the attendant standard analysis forbids Cartesian products $\mathbb{R}^\cup\times\mathbb{F}$ as given above. What, then, shall we do with $\mathbb{F}$? Undefined definitions beg for definitions. En route to some definitions for the arithmetic of $\{\mathcal{F}_\mathcal{X}\}$, let us examine the process of $\mathbb{R}$'s algebraic construction according to Cauchy sequences. We have assumed $\mathbb{N}$ at the outset of our algebraic constructive process to write

$$\mathbb{N}\longrightarrow\mathbb{N}\cup\{0\}\longrightarrow\mathbb{Q}\longrightarrow C_\mathbb{Q}\longrightarrow C_\mathbb{Q}^{\mathbf{AB}}\quad.$$

If we complement the undefined definitions for the arithmetic of immeasurable numbers such that they obey the arithmetic of the natural numbers on the rescaled chart whose $1\in\mathbb{N}_\mathbb{T}$ is on the scale of $\mathcal{F}_0\in\mathbb{R}$ in a "smaller" chart, then the process leading to the mystery set $\mathbb{F}$ becomes

$$\mathbb{N}\longrightarrow\mathbb{N}\cup\{0\}\longrightarrow\mathbb{Q}\longrightarrow C_\mathbb{Q}\longrightarrow C_\mathbb{Q}^{\mathbf{AB}}\longrightarrow\mathbb{N}_\mathbb{T}\quad,$$

Having constructed a bigger version of $\mathbb{N}$, we can use it to construct a transfinite chart without having to assume a second transfinite copy of the naturals with which to begin another constructive process for some $\mathbb{R}_\mathbb{T}$. We can continue to construct "bigger and bigger" copies of $\mathbb{R}$ forever with scale factors greater than or equal to any $n\in\mathbb{N}$ and we will never require any intuitive suppositions beyond the existence of the $n\in\mathbb{N}$ which we have already supposed. We could rescale bigger and bigger forever and we will never fail to have a resultant set which is infinitely small with respect to $\mathbb{T}$. Even if we chose the scale factor $\aleph_{\widehat{\infty}}$, we could make use of the notation being such that

$\aleph_{(\aleph_{\mathcal{X}})} = \mathcal{X}\cdot\widehat{\infty}^2$ to preserve the infinitely-larger-ness of $\mathbb{T}$ as

$$\mathbb{T} = (-\aleph_{\aleph_{\aleph_{\dots}}}, \aleph_{\aleph_{\aleph_{\dots}}}) = (-\widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\cdots}}}}, \widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\widehat{\infty}^{\cdots}}}}) \quad .$$

Of course, the best way to deal with $\mathbb{T}$ would be the simple distinction $\infty > \widehat{\infty}$ but there is no need for such distinction within real analysis. $\mathbb{T}$ lives only in transfinite analysis.

Due to the absorptive properties of geometric infinity, conformal parameters such as $\aleph_{\mathcal{X}}$ never change the Euclidean conceptual component underlying everything: $\mathbb{R} = (\infty, \infty)$. For this reason, the scale that we assign to any particular copy of $\mathbb{R}$ will never disrupt the configuration of an infinite line. It is always true that the interval $(-\widehat{\infty}, \widehat{\infty})$ can be shown to exist as a conformal chart over some finite subset of $(-\infty, \infty)$. $\mathbb{N}_{\mathbb{T}}$ is just another copy of $\mathbb{N}$ with a rescaled unit increment so it must be that $\mathbb{N}_{\mathbb{T}} = \mathbb{N}$. Since the scale of the two sets is different, we cannot allow them to interact by arithmetic because that would require $\mathbb{N}_{\mathbb{T}} \neq \mathbb{N}$. If we took $\mathbb{N}_{\mathbb{T}} \neq \mathbb{N}$, then there would be an implied preferred scale for $\mathbb{R}$ when no such scale exists. So, we have that which was assumed at the outset of the constructive process as the output at the end of same:

$$\mathbb{N} \longrightarrow \mathbb{N} \cup \{0\} \longrightarrow \mathbb{Q} \longrightarrow C_{\mathbb{Q}} \longrightarrow C_{\mathbb{Q}}^{\mathbf{AB}} \longrightarrow \mathbb{N} \quad .$$

We will not allow arithmetic among the measurables and immeasurables so there is no reason to retain the subscript label $\mathbb{T}$ which reminds us that the scale of one instance of $\mathbb{N}$ differs from another. When that matters, we call the larger numbers $\mathcal{F}(n)$ where all the problems associated with the (n) notation have been done away with by Axiom 7.5.8 forbidding the given forms of $\mathbb{R}^{\cup} \times \mathbb{F}$. Now the label (n) tells us which (extended) natural number $n \in \mathbb{N}_{\mathbb{T}}$ is each $\mathcal{F}(n) \in \mathbb{R}$. Now we have a ready-made set of arithmetic axioms for $\{\mathcal{F}_{\mathcal{X}}\}$: they should obey the arithmetic laws of natural numbers with themselves.

Axiom 7.5.21 The arithmetic operations of immeasurable real numbers with themselves are the arithmetic operations of the extended natural numbers $\mathbb{N}_{\infty}$ such that there exists a one-to-one correspondence $\mathbb{R} \ni \mathcal{F}(n) \to n \in \mathbb{R}_{\mathbb{T}}$. The arithmetic operations take two immeasurable $\mathcal{F}_{\mathcal{X}} \in \mathbb{R}$ and return a cut in another number line which is an identical but distinctly labeled copy of $\mathbb{R}$. In particular, we have

$$\begin{aligned} \mathcal{F}(n) + \mathcal{F}(m) &= \mathcal{F}(n+m) &\longrightarrow\quad & n+m \in \mathbb{R}_{\mathbb{T}} \\ \mathcal{F}(n) \cdot \mathcal{F}(m) &= \mathcal{F}(n \cdot m) &\longrightarrow\quad & n \cdot m \in \mathbb{R}_{\mathbb{T}} \\ \mathcal{F}(n) \div \mathcal{F}(m) &= \mathcal{F}(n \div m) &\longrightarrow\quad & n \div m \in \mathbb{R}_{\mathbb{T}} \quad . \end{aligned}$$

Remark 7.5.22 By now, we have avoided the least upper bound problem with $\{\mathcal{F}_{\mathcal{X}}\} \subset \mathbb{R}$, but it remains to show that $\{\mathcal{F}_{\mathcal{X}}\}$ satisfies the Archimedes property. There is *most likely* no symbolic representation of the statement, "Magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another," which does not rely on arithmetic. Therefore, it will be required that we define arithmetic operations for non-arithmatic numbers with themselves. Axiom 7.5.8 only forbids arithmetic between measurable and immeasurable numbers so we are free to define arithmetic operations for the immeasurable non-arithmatic numbers with themselves, as in Axiom 7.5.21.

In constructing $\mathbb{R}$, we have assumed $\mathbb{N}$ as a discrete set without requiring that the numbers are regularly spaced cuts along the real line. The idea of one apple or 52 apples does not require the presupposition of the existence of a 1D Hausdorff space extending infinitely far in both directions. It suffices to grant that if whole apples are had, then we can have a few of them or a lot. We have assumed $n \in \mathbb{N}$ as a discrete set and then combined it with the left endpoint of a geometric line segment to construct the cuts in $\mathbb{R}$ in terms of Cauchy sequences. Going back to Definition 2.1.3 we see that the real line is only a line with an appended label "real" so the requirement $\mathbb{R} = \mathbb{R}_{\mathbb{T}}$ simply augments the label we put on the line. Now we have "this real line" and "that real line" with a specification of the relative scale of the unit of Euclidean distance in either of them. Among two number lines, therefore, we may call one "the real line" and the other "the big real line" but it necessarily follows that "the real line" is identically the big real line in the convention where we attach the simple label "real" to what would be "the small real line." Cuts in the big real line should be constructed from the $\mathbb{N}_{\mathbb{T}}$ output at the end of the construction of the first real line.

In defining $\mathcal{F}_{\mathcal{X}} = \mathcal{F}(n) \to n$ as cuts in the big real line, we avoid all of the arithmetic problems associated with the (n) labeling scheme. Although there are irreparable contradictions with the idea that $\frac{1}{2}\mathcal{F}_0 = \frac{1}{2}\mathcal{F}(1)$ should be a cut $x \in \mathbb{R}_0$, there is no problem when $\frac{1}{2}\mathcal{F}(1)$ is cut in the separate "big" real line where the quantity is expressed as $\frac{1}{2}\mathcal{F}(1) \to \frac{1}{2} \cdot 1 = \frac{1}{2}$. The $\mathcal{F}(n)$ are the underlying discrete elements upon which the cuts in the big real line are defined according to Cauchy equivalence classes. The label (n) marks which $\mathcal{F}_{\mathcal{X}}$ in the real line is which n in the big real line. Furthermore, since all copies of $\mathbb{R}$ are the same, meaning that they differ by nothing more than a label, there is no strict requirement to define a second line. It will suffice to two define two charts: the Euclidean chart and "the big Euclidean chart." Since we have taken $(-\widehat{\infty}, \widehat{\infty})$ as an embedded chart in $(-\infty, \infty)$, there is no problem assuming another chart whose linear extend exceeds $(-\widehat{\infty}, \widehat{\infty})$.

If we began with $\mathbb{N}$ as cuts in $\mathbb{R}$, then that would necessarily be a disfavorably circular line of reasoning. We have supposed the preexistence of $\mathbb{N}$ as being separate from $\mathbb{R}$ and now we have a nice explanation for what are these abstract n which show up as regularly spaced cuts in $\mathbb{R}$ once we have it built. $\mathbb{N} \subset \mathbb{R}$ are the $\mathcal{F}(n)$ in the "small real line" which we may as well label $\mathbb{R}_{\mathbb{T}^{-1}}$. While this reasoning is nearly circular, it is not at all circular because it does still require some initial supposition of the existence of $\mathbb{N}$. Once that is supposed, however, we may construct an infinite tier of increasingly large Euclidean charts on $\mathbb{R}$ without any further supplemental abstractions. Call them $\mathbb{R}_{\mathbb{T}^n}$. After defining an infinite succession of such charts, we can pick any of them as "$\mathbb{R}$" since it is given that $\mathbb{R}_{\mathbb{T}}$ differs from $\mathbb{R}$ only in its label.

Main Theorem 7.5.23 *Immeasurable numbers $\mathcal{F}_{\mathcal{X}} \in \mathbb{R}$ obey the Archimedes property of real numbers.*

Proof. Definition 6.3.9 gives the Archimedes property of real numbers as

$$\forall x, y \in \mathbb{R} \quad \text{s.t.} \quad x < y \quad \exists z_1, z_2 \in \mathbb{R}^+ \quad \text{s.t.} \quad z_1 x > z_2 y \quad .$$

As in the proof of Main Theorem 6.3.10, we will divide this proof into cases of the Archimedean statement $x < y$.

• ($x \in \mathbb{R}^{\mathcal{X}}$, $y = \mathcal{F}_{\mathcal{Y}}$, $\mathcal{Y} \geq \mathcal{X}$) We have $x < y$ so choose two multipliers $z_1 = \frac{\mathcal{Z}}{\mathcal{X}}$ such that $\mathcal{Y} < \mathcal{Z} < 1$, and choose $z_2 = \mathcal{F}_0 = \mathcal{F}(1)$ where $\mathcal{F}_0 \in \mathbb{R}$ is the multiplicative identity $1 \in \mathbb{R}_{\mathbb{T}}$ for the arithmetic of the immeasurables granted by Axiom 7.5.21. Then

$$\frac{\mathcal{Z}}{\mathcal{X}}\big(\aleph_{\mathcal{X}} + b\big) = \aleph_{\mathcal{Z}} + \frac{b\mathcal{Z}}{\mathcal{X}} \quad > \quad \mathcal{F}_{\mathcal{Y}} \quad .$$

• ($x = \mathcal{F}_{\mathcal{X}}$, $y \in \mathbb{R}^{\mathcal{Y}}$, $\mathcal{Y} \geq \mathcal{X}$) Choose $z_1 = \mathcal{F}_0$ and $z_2 = \frac{\mathcal{Z}}{\mathcal{Y}}$ such that $\mathcal{Z} \leq \mathcal{X}$. Then

$$\mathcal{F}_{\mathcal{X}} \quad > \quad \frac{\mathcal{Z}}{\mathcal{Y}}\big(\aleph_{\mathcal{Y}} + b\big) = \aleph_{\mathcal{Z}} + \frac{b\mathcal{Z}}{\mathcal{Y}} \quad .$$

• ($x = \mathcal{F}_{\mathcal{X}}$, $y = \mathcal{F}_{\mathcal{Y}}$, $\mathcal{Y} \geq \mathcal{X}$) For the purposes of the arithmetic of the immeasurables, there is no problem using the $\mathcal{F}_{\mathcal{X}} = \mathcal{F}(n)$ notation. In this case, we take the $\mathcal{F}(n) = n \in \mathbb{N}_{\mathbb{T}}$ as a discrete exterior set upon which the cuts in the line $\mathbb{R}_{\mathbb{T}}$ are constructed. All the problems demonstrated in and around Section 7.3 relied upon assumptions of the form

$$\begin{aligned} x + y &= f_+(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ x \cdot y &= f_\times(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ x \div y &= f_\div(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad , \end{aligned}$$

which we find are now better written in the respective forms of

$$f(x, y) : \mathbb{R}^\cup \times \mathbb{R}^\cup \to \mathbb{R} \quad .$$

Now, Axiom 7.5.21 grants forms following

$$f(\mathcal{F}_{\mathcal{X}}, \mathcal{F}_{\mathcal{Y}}) : \mathbb{F} \times \mathbb{F} \to \mathbb{R}_{\mathbb{T}} \quad ,$$

where cuts in $\mathbb{R}_{\mathbb{T}}$ are Euclidean magnitudes "of a different kind" than those in $\mathbb{R}$. (Recall that Axiom 7.5.8 forbids the domain $\mathbb{R}^\cup \times \mathbb{F}$ for any of f_+, $f_\times$, or $f_\div$.)

In Book 5 of *The Elements*, Definition 3, Euclid wrote, "A ratio is a certain type of condition with respect to size of two magnitudes of the same kind." By defining the non-arithmatic real numbers of one kind as simultaneous natural numbers of another kind, we avoid all of the problems associated with the $\mathcal{F}(n)$ notation: the notation $x = \mathcal{F}_\mathcal{X}$ tells us to treat x as a cut in $\mathbb{R}$ and $x = \mathcal{F}(n)$ tells us to treat it as a cut in $\mathbb{R}_\mathbb{T}$. In either case, x is a cut in the real number line. Therefore, regarding the Archimedes property for $x = \mathcal{F}_\mathcal{X}$ and $y = \mathcal{F}_\mathcal{Y}$, let

$$\mathcal{F}_\mathcal{X} = \mathcal{F}(n) \quad , \quad \text{and} \qquad \mathcal{F}_\mathcal{Y} = \mathcal{F}(m) \quad ,$$

so that $\mathcal{F}(n) < \mathcal{F}(m)$. Choose $z_1 = \mathcal{F}(k)$ such that $nk > m$ and $z_2 = \mathcal{F}(1)$ so that Axiom 7.5.21 gives

$$\mathcal{F}(k) \cdot \mathcal{F}(n) = \mathcal{F}(kn) \quad > \quad \mathcal{F}(m) \quad .$$

We have demonstrated the main cases and conclude the proof with an assumption that the other cases follow directly. Non-arithmatic immeasurable numbers satisfy the ancient Archimedes property of real numbers. ✐

Remark 7.5.24 Due to the explicit reliance on multiplication by $\aleph_n$ in the Archimedes property of 1D transfinitely continued real numbers (Axiom 6.3.12) we cannot directly demonstrate the compliance due to the inadmissible domain of

$$f_\times(\mathcal{F}_\mathcal{X}, \aleph_n) : \mathbb{F} \times \mathbb{R}^\cup \to \mathbb{R}_\mathbb{T} \quad .$$

To the extent that Axiom 6.3.12 was proposed only to simply the demonstration of the compliance of numbers in the neighborhood of infinity with the Archimedes property, we could more precisely call this axiom the Archimedes property of 1D transfinitely continued arithmatic real numbers.

Theorem 7.5.25 *If $\mathbb{R}$ and $\mathbb{R}_\mathbb{T}$ are taken as every value of two charts x and $x_\mathbb{T}$ on the same line, then the two charts cannot share an origin.*

Proof. The radius of the neighborhood of the origin $\mathbb{R}_0$ is one half the length of any element of $\{\mathbb{R}^\mathcal{X}\}$. The origin of the x chart lies one half as far to the left of $\mathcal{F}(1) = 1 \in \mathbb{N}_\mathbb{T}$ as $\mathcal{F}(2) = 2 \in \mathbb{N}_\mathbb{T}$ lies to the right of it. Now that we have given arithmetic for the immeasurables, we may preserve the notion of Euclidean distance between immeasurable with the ordinary Euclidean metric. The big metric $d_\mathbb{T}(x_\mathbb{T}, y_\mathbb{T}) = |y_\mathbb{T} - x_\mathbb{T}|$ requires that distance is uniform over the line. Therefore, the cut $0 \in \mathbb{R}$ cannot have a simultaneous algebraic representation as $0 \in \mathbb{R}_\mathbb{T}$. Such a representation would require a non-Euclidean notion of distance rendering the line in question something other than a number line as given by Definition 2.1.2. With or without a metric, numbers must be spaced evenly along a number line for the purposes of Euclidean geometry. ✐

§7.6 The Topology of the Real Number Line

The thesis of the fractional distance analysis presented here has to been preserve the Euclidean geometric construction of $\mathbb{R}$ through an algebraic construction which does not preclude the existence of the neighborhood of infinity. We began with Axiom 2.1.7 stating that real numbers are represented in algebraic interval notation as $\mathbb{R} = (-\infty, \infty)$. This axiom is totally equivalent to a requirement that $\mathbb{R}$ has the usual topology. In this section, we will define two different topologies on $\mathbb{R}$: the usual topology and another which we call the fractional distance topology.

Definition 7.6.1 A topology on a set S is a collection $\mathcal{T}$ of open subsets of S with the following properties.

- $\mathcal{T}$ contains S and $\varnothing$.
- $\mathcal{T}$ contains the union of any of the elements of $\mathcal{T}$: $\bigcup \tau_k \in \mathcal{T}$.
- $\mathcal{T}$ contains the finite intersection of the elements of $\mathcal{T}$: if $n < \infty$, then $\bigcap\limits_{k=1}^{n} \tau_k \in \mathcal{T}$.

The open sets in $\mathcal{T}$ are called the basis of the topology; the topology is the set of all unions of the sets in its basis. Together, the pair $(S, \mathcal{T})$ is called a topological space.

Definition 7.6.2 A basis $\mathcal{B}$ for a topology $\mathcal{T}$ on S is a set $\{\mathcal{B}_k\}$ of subsets $\mathcal{B}_k \subset S$ such that (i) for every $s \in S$ there is at least one basis element which contains s, and such that (ii) if $S \in \mathcal{B}_1 \cap \mathcal{B}_2$, then there exists $\mathcal{B}_3 \subseteq \mathcal{B}_1 \cap \mathcal{B}_2$ such that $s \in \mathcal{B}_3$.

Definition 7.6.3 The *usual* basis $\mathcal{B}_0$ for the usual topology on $\mathbb{R}$ is the collection of all 1D open intervals such that

$$\mathcal{B}_0 = \Big\{(a,b) \;\Big|\; [a],[b] \subset C_{\mathbb{Q}},\ a < b\Big\} \quad ,$$

The topology generated by $\mathcal{B}_0$ is called $\mathcal{T}_0$.

Theorem 7.6.4 *$\mathcal{B}_0$ is not a basis for a topology on $\mathbb{R}$.*

Proof. Definition 7.6.2 requires that

$$\forall x \in \mathbb{R} \quad \exists (a,b) \in \mathcal{B}_0 \quad \text{s.t.} \quad x \in (a,b) \quad .$$

Consider $x = \aleph_{0.5} \in \mathbb{R}$. We have

$$\forall a = [a] \quad \forall b = [b] \quad \text{s.t} \quad [a],[b] \in C_{\mathbb{Q}} \quad \exists n \in \mathbb{N} \quad \text{s.t.} \quad a, b < n \quad .$$

It follows that there is no interval $(a,b) \in \mathcal{B}_0$ which contains $x > n$. This proves the theorem. ∎

Theorem 7.6.5 *The pair* $(\mathbb{R}_0, \mathcal{T}_0)$ *is a topological space when* $\mathcal{T}_0$ *is the topology generated by* $\mathcal{B}_0$.

Proof. Proof follows from Definitions 7.6.1, 7.6.2, and 7.6.3.

Remark 7.6.6 "Having the usual topology" is exactly equal to Axiom 2.1.7 granting that $\mathbb{R} = (-\infty, \infty)$. If all of $\mathbb{R}$ is axiomatically taken to reside within the neighborhood of the origin, then $(\mathbb{R}, \mathcal{T}_0)$ is a well-defined topological space and $\mathcal{T}_0$ is the usual topology on $\mathbb{R}$. Since we have included the neighborhood of infinity in $\mathbb{R}$, rightly, we need to adjust the basis of the usual topology to reflect its inclusion. For this reason, we cannot construct (generate) the usual topology with its *usual* basis $\mathcal{B}_0$. Moving in that direction, now we will give some redundant axioms which clarify exactly what we need to put into the open sets constituting the fractional distance basis of the usual topology $\mathcal{T}_{\text{U}}$ on $\mathbb{R}$ if we are to avoid an undue and contradictory truncation of the neighborhood of infinity.

Axiom 7.6.7 ***The fundamental axiom of geometric construction***. Non-negative real numbers are algebraic representations of points in the infinitely long line segment **AB**, *i.e.*:

$$\mathbb{R}^+ = \left\{ x \;\middle|\; x \in X \in \mathbf{AB},\ 0 < x < \widehat{\infty} \right\} \quad .$$

Axiom 7.6.8 ***The fundamental axiom of algebraic construction***. The topology of the real number line is the usual one.

Remark 7.6.9 Axiom 7.6.7 posits the existence of every measurable number $x \in \mathbb{R}^\cup$. Axiom 7.6.8 posits the existence of the immeasurables $x \in \mathbb{F}$ so that there are no gaps in $(-\infty, \infty)$ which would prevent us from taking the usual topology $\mathcal{T}_{\text{U}}$ as the set of all unions of $(a, b) \subset (-\infty, \infty)$. Without granting $\{\mathcal{F}_\mathcal{X}\} \subset \mathbb{R}$, intervals of the form $(\aleph_\mathcal{X}, \aleph_{\mathcal{X}+\delta})$ with $\delta > 0$ would not be connected subsets of $\mathbb{R}$. They would be disconnected at the immeasurable extrema of the $\mathbb{R}^\mathcal{X}$ neighborhoods. They would not be intervals at all be because intervals are necessarily connected.

Definition 7.6.10 The fractional distance topology on $\mathbb{R}^\cup$ is $\mathcal{T}_{\text{FD}}$ generated by a basis $\mathcal{B}_{\text{FD}} = \mathcal{B}_\mathcal{X} \cup \mathcal{B}_\infty$ such that

$$\mathcal{B}_\mathcal{X} = \left\{ (\aleph_\mathcal{X} + a, \aleph_\mathcal{X} + b) \;\middle|\; [\mathcal{X}], [a], [b] \subset C_\mathbb{Q},\ a < b,\ -1 < \mathcal{X} < 1 \right\}$$

$$\mathcal{B}_\infty = \left\{ (\pm\widehat{\infty} \mp a, \pm\widehat{\infty} \mp b) \;\middle|\; [a], [b] \subset C_\mathbb{Q},\ a > b > 0 \text{ if } +\widehat{\infty},\ 0 < a < b \text{ if } -\widehat{\infty} \right\} .$$

Theorem 7.6.11 *The topological space* $(\mathbb{R}^\cup, \mathcal{T}_{FD})$ *satisfies Axiom 7.6.7.*

Proof. Proof follows from Definition 7.6.10. Every x required by Axiom 7.6.7 appears in the basis $\mathcal{B}_{\text{FD}}$ of the topology $\mathcal{T}_{\text{FD}}$.

Remark 7.6.12 In the present conventions, $(\mathbb{R}^{\cup}, \mathcal{T}_{\text{FD}})$ is a well-defined topological space but $(\mathbb{R}, \mathcal{T}_{\text{FD}})$ is not. If so desired, however, one could axiomatize $(\mathbb{R}, \mathcal{T}_{\text{FD}})$ as a topological space such that every real number is measurable. This would necessarily overwrite the fundamental axiom of algebraic construction but one might find such and axiom more useful in certain circumstances. Real numbers existed for thousand of years before topology was conceived so we cannot assign the same sacred status to the usual topology that we give to other restrictions on $\mathbb{R}$ such as the Archimedes property or $1 + 1 = 2$.

Definition 7.6.13 The fractional distance basis $\mathcal{B}_{\text{U}}$ for the usual topology $\mathcal{T}_{\text{U}}$ on $\mathbb{R}$ is the collection of all 1D open intervals such that

$$\mathcal{B}_{\text{U}} = \Big\{ (a,b) \ \Big| \ a, b \in \overline{\mathbb{R}}, \ a < b \Big\} \quad .$$

Remark 7.6.14 The *usual* basis $\mathcal{B}_0$ for the usual topology (in its incarnation $\mathcal{T}_0$, as per Definition 7.6.3) assumes that all real numbers are identically Cauchy equivalence classes of rationals $[x] \subset C_{\mathbb{Q}}$. The fractional distance basis $\mathcal{B}_{\text{U}}$ for the usual topology $\mathcal{T}_{\text{U}}$ takes into account the current convention that real numbers are either extended equivalence classes $[x] \subset C_{\mathbb{Q}}^{\mathbf{AB}}$ or extended Dedekind partitions $x = (L, R)$.

Theorem 7.6.15 *The topological space* $(\mathbb{R}, \mathcal{T}_U)$ *satisfies Axiom 7.6.8.*

Proof. Proof follows from Definition 7.6.13. Every x required by Axiom 7.6.8 appears in the basis sets of $\mathcal{T}_{\text{U}}$ because

$$x \in \overline{\mathbb{R}} \qquad \Longleftrightarrow \qquad \pm x \in \Big\{ \mathbb{R}^{\cup} \cup \mathbb{F} \cup \widehat{\infty} \Big\} \quad .$$

§8 The Riemann Hypothesis

§8.1 The Riemann Zeta Function

The Riemann hypothesis dates to Riemann's 1859 paper [16]. Since the axioms of a complete ordered field date, at earliest, to Hilbert's 1899 paper [4], it would be improper to claim that the Riemann hypothesis is formulated in terms of the ordered field definition of $\mathbb{R}$. Likewise, Dedekind's partition definition [3] and Cantor's definition of real numbers as equivalence classes of rationals [2] date to a pair of 1872 papers so the Riemann hypothesis cannot be understood as being phrased in the language of real numbers as Dedekind cuts or Cauchy sequences. The topological space as a mathematical concept did not exist until well into the 20th century so it would be similarly absurd to claim that the Riemann's hypothesis is formulated in terms of the usual topology on $\mathbb{R}$. While we cannot directly show what definition of $\mathbb{R}$ Riemann had in mind when formulating his hypothesis, we may glean very much from the plain fact that Riemann made no comment or nod toward any definition of $\mathbb{R}$ whatsoever. This should be taken to mean that Riemann assumed his definition of $\mathbb{R}$ would have been absolutely, unambiguously known *a priori* to his intended audience. The only possible definition which might have been available to satisfy this condition in 1859 was Euclid's definition of real numbers as geometric magnitudes. Indeed, Riemann's program of Riemannian geometry is a direct extension of Euclidean geometry so, to a very high degree, this qualitatively supports the notion that Riemann had in mind the cut-in-a-number-line definition of $\mathbb{R}$ given by Euclid in *The Elements*.

When one examines *The Elements* [1], the very many diagrams, definitions, and postulates make it exceedingly obvious that Euclid's definition of a real number as a magnitude, one having a proportion and ratio with respect to all other magnitudes of the same kind, is exactly the one given here in Definition 2.1.4. That definition gives

$$\mathbb{R}\setminus x=(-\infty,x)\cup(x,\infty)\quad,$$

as an alternative identical formulation of the Euclidean statement

$$x\in\mathbb{R}^+\qquad\Longleftrightarrow\qquad(0,\infty)=(0,x]\cup(x,\infty)\quad.$$

It is reasonable to conclude that Riemann formulated his hypothesis with it in mind that any definition of $\mathbb{R}$ consistent with the Euclidean magnitude would be sufficient. The domain of $\zeta(z)$, namely $\mathbb{C}$, would be constructed from two orthogonal copies of $\mathbb{R}$, one of them having the requisite phase factor i. Rather than the underlying definition of $\mathbb{R}$, the object of relevance in the Riemann hypothesis should be the behavior of $\zeta(z)$ at various z.

Definition 8.1.1 Complementing Definition 6.2.8 which gave $z\in\mathbb{C}$ as any $z=x+iy$ such that $x,y\in\mathbb{R}$, we define the complex neighborhood of the origin as

$$\mathbb{C}_0=\big\{x+iy\;\big|\;x,y\in\mathbb{R}_0\big\}\quad.$$

Definition 8.1.2 The arithmatic subset of the complex plane $\mathbb{C}^\cup\subset\mathbb{C}$ is such that

$$\mathbb{C}^\cup=\big\{x+iy\;\big|\;x,y\in\mathbb{R}^\cup\big\}\quad.$$

Definition 8.1.3 For $x\in\mathbb{R}$, the Dirichlet series is

$$\mathfrak{D}(x)=\sum_{n=1}^{\infty}\frac{1}{n^x}\quad.$$

This series converges absolutely for all $x \in \mathbb{R}^{\cup}$ such that $x > 1$. Formally, this is a special case of Dirichlet series but we will simply call it *the* Dirichlet series.

Definition 8.1.4 For $z \in \mathbb{C}$, the Dirichlet form of the Riemann ζ function is

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad .$$

This series converges absolutely for all $z \in \mathbb{C}$ such that $\text{Re}(z) > 1$.

Remark 8.1.5 The Riemann hypothesis primarily concerns the behavior of $\mathfrak{D}$ continued onto the region of $\mathbb{C}$ with real parts exceeding the domain of convergence of $\mathfrak{D}$, *i.e.*: the region $(-\infty, 1]$. Riemann's famous functional equation [16–28], given here in Definition 8.1.6, enforces the absolute convergence of ζ on regions of $\mathbb{C}$ whose real parts exceed the domain of convergence of $\mathfrak{D}$. In Theorem 8.1.8, we will prove that $\mathfrak{D}(x)$ converges even for $x \in \mathbb{F}$. Then it will follow trivially that the convergent behavior of $\mathfrak{D}$ at non-arithmatic $x \in \mathbb{R} \setminus \mathbb{R}^{\cup}$ carries over to ζ at non-arithmatic $z \in \mathbb{C} \setminus \mathbb{C}^{\cup}$. This follows because (i) Definition 8.1.4 sets the ζ function exactly equal to $\mathfrak{D}(x)$ when $\text{Re}(z) > 1$ and $\text{Im}(z) = 0$, and (ii) the functional form given in Definition 8.1.6 is exactly equal to the Dirichlet form of ζ on its domain of convergence.

Definition 8.1.6 Riemann's functional equation for the absolutely convergent analytic continuation of $\mathfrak{D}$ is

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

In the region $\text{Re}(z) > 1$, this form of ζ is exactly equal the Dirichlet form given in Definition 8.1.4.

Remark 8.1.7 Under the classical assumption that $\mathbb{C} = \mathbb{C}_0$, it is said that Riemann's functional equation converges absolutely on $\mathbb{C} \setminus Z_1$ where

$$Z_1 = z(x,y) = z(r,\theta) = 1 \quad ,$$

is such that $\zeta(Z_1) \notin \mathbb{C}$. Presently, the many historical demonstrations of the convergence of Riemann's functional equation for ζ are only valid on $\mathbb{C}^{\cup} \setminus Z_1$ because the immeasurables have not been previously considered in this context. Since $\mathfrak{D}(\mathcal{F}_{\mathcal{X}})$ is not yet clarified, we can only say with certainty that Riemann's functional equation is absolutely convergent on $\mathbb{C}^{\cup} \setminus Z_1$. However, without regard for the behavior of ζ on $\mathbb{C} \setminus \mathbb{C}^{\cup}$, we already have all the tools needed to determine its behavior on $\mathbb{C}^{\cup}$. These tools are sufficient to close the book on the Riemann hypothesis as an open question.

On the other hand, it is good for thoroughness to understand the behavior of $\mathfrak{D}(x)$ at $x \in \mathbb{F}$. It is understood that $\mathfrak{D}(x)$ is absolutely convergent on the ray $x > 1$ which includes immeasurable, non-arithmatic numbers $\{\mathcal{F}_\mathcal{X}\} \subset (1,\infty)$ but we have not yet given a concise definition for $x^{\mathcal{F}_\mathcal{X}}$. The absolute convergence of $\mathfrak{D}$ everywhere on this ray is usually framed in the context of every real number being less than some natural number so we should give due consideration to the numbers in the neighborhood of infinity. It follows from the arithmetic axioms that $\mathfrak{D}(x) = 1$ for any $x \in \mathbb{R}^\cup \setminus \mathbb{R}_0$ but the case of

$$\mathfrak{D}(\mathcal{F}_\mathcal{X}) = \sum_{n=1}^{\infty} \frac{1}{n^{\mathcal{F}_\mathcal{X}}} \quad ,$$

remains to be clarified. We have required with Axiom 7.5.8 that arithmetic operations between measurable and immeasurable numbers are not defined but the definition of exponentiation is such that $n^{\mathcal{F}_\mathcal{X}}$ involves arithmetic operations between natural numbers only:

$$n^{\mathcal{F}_\mathcal{X}} = \underbrace{n \cdot n \cdot n \cdot n \cdot n \cdot n \dots n}_{\text{product of } \mathcal{F}_\mathcal{X} \; n\text{s}} \quad .$$

Raising a natural number to the power of an immeasurable number only makes a call to the immeasurables when one wants to count the number of products of naturals. Therefore, there is no reason to discount $n^{\mathcal{F}_\mathcal{X}}$ as an undefined operation pursuant to Axiom 7.5.8.

Theorem 8.1.8 *If $0 \leq \mathcal{X} < 1$, then the quantity $\mathfrak{D}(\mathcal{F}_\mathcal{X})$ is equal to one.*

Proof. The Dirichlet series is

$$\mathfrak{D}(x) = \sum_{n=1}^{\infty} n^{-x} = \sum_{n=1}^{\infty} \left(e^{\ln n}\right)^{-x} = \sum_{n=1}^{\infty} e^{-x \ln n} \quad ,$$

so we have

$$\frac{d}{dx}\mathfrak{D}(x) = \sum_{n=1}^{\infty} -\ln n e^{-x \ln n} \quad .$$

The derivative of $\mathfrak{D}(x)$ is non-positive on $x > 1$ due to the overall minus sign. Since the function never increases on the ray, and since the arithmetic axioms give

$$x_0 \in \left\{\mathbb{R}^\mathcal{X}\right\} \quad \implies \quad \mathfrak{D}(x_0) = \sum_{n=1}^{\infty} \frac{1}{n^{\aleph_\mathcal{X}+b}} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{\aleph_\mathcal{X}+b}} = 1 \quad .$$

it follows that $\mathfrak{D}(x)$ is monotonic decreasing or constant on the ray. Noting that every number $\aleph_\mathcal{X} \in (1,\infty)$ is such that $\mathcal{X} > 1$, $\mathfrak{D}(x)$ is constant on $(\aleph_\mathcal{X}, \aleph_\mathcal{Y}) \subset (1,\infty)$ because $\mathfrak{D}(\aleph_\mathcal{X} - 1) = 1$ and the function never increases on the ray. Since $\mathcal{F}_\mathcal{X} \in (\aleph_\mathcal{X}, \aleph_\mathcal{Y})$ and $\mathfrak{D}$ is constant on this interval, it follows that

$$\mathcal{X} > 0 \quad \implies \quad \mathfrak{D}(\mathcal{F}_\mathcal{X}) = 1 \quad .$$

It follows, therefore, that

$$n > 1 \quad , \quad \mathcal{X} > 0 \quad \implies \quad \frac{1}{n^{\mathcal{F}_\mathcal{X}}} = 0 \quad .$$

To complete the proof of the present theorem, we need to show

$$n > 1 \quad \implies \quad \frac{1}{n^{\mathcal{F}_0}} = 0 \quad .$$

We will prove this remaining case of $\mathcal{F}_0$ by contradiction. If $n^{-\mathcal{F}_0} > 0$, then

$$\exists m \in \mathbb{N} \quad \text{s.t.} \quad \frac{1}{n^{\mathcal{F}_0}} > \frac{1}{m} \quad \implies \quad n^{\mathcal{F}_0} < m \quad .$$

Since $n \in \mathbb{N}$, the condition $n > 1$ is equivalent to the condition $n \geq 2$. Since $\mathcal{F}_0 > 1$, it follows that

$$\frac{1}{n^{\mathcal{F}_0}} < \frac{1}{2} \quad \implies \quad m > 2 \quad .$$

Every such m maybe expressed as $m = n^k$ for some $k \in \mathbb{R}_0$. Then

$$n^{\mathcal{F}_0} < m \quad \implies \quad n^{\mathcal{F}_0} < n^k \quad \implies \quad \mathcal{F}_0 < k \quad .$$

Since $k \in \mathbb{R}_0$, we have obtained a contradiction. The theorem is proven.

Remark 8.1.9 In Theorem 6.2.5, we used the result

$$\big(\aleph_\mathcal{X} \pm k\big)! = \aleph_{\aleph_{\aleph_{\dots}}} = \infty \quad ,$$

to prove that the big exponential function E^x is equal to the regular exponential function e^x for any $x \in \mathbb{R}_0$.

§8.2 Non-trivial Zeros in the Critical Strip

In this section, we will prove the negation of the Riemann hypothesis.

Remark 8.2.1 The Riemann ζ function is holomorphic on $\mathbb{C} \setminus Z_1$. It is a well-known property of holomorphic functions that their zeros are isolated on a domain or else the function is constant on that domain. However, this property relies on the implicit axiom that all pairs of points (z_1, z_2) in any subdomain $D \subset \mathbb{C}$ are such that the distance between them is in $d(z_1, z_2) \in \mathbb{R}_0$. When we do not take this implicit axiom, further specification is required. The property becomes the following. If the zeros of a holomorphic function are not isolated, then the function is constant everywhere on a disc of radius $r_0 \in \mathbb{R}_0$ about any of the non-isolated zeros.

Proposition 8.2.2 If (i) f is a holomorphic function defined everywhere on an open connected set $D \subset \mathbb{C}$, and (ii) there exists more than one $z_0 \in D$ such that $f(z_0) = 0$, then f is constant on D or the set containing all $z_0 \in D$ is totally disconnected.

Refutation. This proposition is usually proven by a line of reasoning starting with the following. By the holomorphism of f and the property $f(z_0) = 0$, we know there exists a convergent Taylor series representation of $f(z)$ for all $|z - z_0| < r_0$ with $r_0 \in \mathbb{R}$. Here, the proposition immediately fails pseudo-trivially because we can select $r_0 \in \{\mathbb{R}^{\mathcal{X}}\}$ and assume

$$\big|z - z_0\big| > \big(\aleph_{\mathcal{X}} + a\big) \quad ,$$

to show that the Taylor series does not converge when $\mathcal{X} > 0$. We have

$$f(z) = f(z_0) + f'(z_0)\big(z - z_0\big) + \frac{f''(z_0)}{2!}\big(z - z_0\big)^2 + ... \quad .$$

The first term in the series vanishes by definition so, therefore, we have by assumption

$$f(z) > f'(z_0)\big(\aleph_{\mathcal{X}} + b\big) + \sum_{n=2}^{\infty} \frac{f^{(n)}(z_0)}{n!}\big(\aleph_{\mathcal{X}} + b\big)^n \quad .$$

The Taylor series expansion of f does not converge in $\mathbb{R}$ for $|z - z_0| \in \mathbb{R}_0^{\mathcal{X}}$. This follows from $(\aleph_{\mathcal{X}} + b)^n > \aleph_1$ for all $n \geq 2$, as per Axiom 5.2.5.

Axiom 8.2.3 If (i) f is a holomorphic function defined everywhere on an open connected set $D \subset \mathbb{C}$, (ii) there exists more than one $z_0 \in D$ such that $f(z_0) = 0$, and (iii) every $p \in D$ is such that $|z_0 - p| \in \mathbb{R}_0$, then f is constant on D or the set containing all $z_0 \in D$ is totally disconnected.

Remark 8.2.4 Various proofs of Axiom 8.2.3 are well-known. They are taken for granted.

Main Theorem 8.2.5 *If $\{\gamma_n\}$ is an increasing sequence containing the imaginary parts of the non-trivial zeros of the Riemann ζ function in the upper complex half-plane, then*

$$\lim_{n \to (\aleph_{\mathcal{X}} + b)} \big|\gamma_{n+1} - \gamma_n\big| = 0 \quad .$$

Proof. To prove the theorem, we will follow Titchmarsh's proof [17] of a theorem of Littlewood [29]. The original theorem is as follows.

"For every large T, $\zeta(s)$ has a zero $\beta + i\gamma$ satisfying

$$|\gamma - T| < \frac{A}{\log\log\log T} \quad ."$$

Note that A is some constant $A \in \mathbb{R}_0$. For proof by contradiction, assume

$$\lim_{n \to (\aleph_{\mathcal{X}}+b)} \left| \gamma_{n+1} - \gamma_n \right| \neq 0 \quad .$$

Then there exists some $m(n)$ and some $a \in \mathbb{R}_0^+$ such that

$$\lim_{m(n) \to (\aleph_{\mathcal{X}}+b)} \left| \gamma_{m(n)+1} - \gamma_{m(n)} \right| > 2a \quad .$$

Let T_n be the average of $\gamma_{m(n)+1}$ and $\gamma_{m(n)}$ so

$$T_n = \frac{\gamma_{m(n)+1} + \gamma_{m(n)}}{2} \quad .$$

Now we have

$$\lim_{T_n \to (\aleph_{\mathcal{X}}+b)} \left| \gamma - T_n \right| > a \quad ,$$

because T_n is centered between the next greater and next lesser γ_n. We have shown that this pair of γ_n are separated by more than $2a$. This contradicts Littlewood's result

$$|\gamma - T_n| < \frac{A}{\log\log\log T_n} \quad , \qquad \text{whenever} \qquad \frac{A}{\log\log\log T_n} < a \quad .$$

The limit $T_n \to \aleph_{\mathcal{X}} + b$ is exactly such a case because

$$\log(\aleph_{\mathcal{X}} + b) = \log(\mathcal{X}\widehat{\infty}) + \log(b) = \log(\mathcal{X})\log(\widehat{\infty}) + \log(b) \quad .$$

If we take $\log(\widehat{\infty}) = \widehat{\infty}$ or $\log(\widehat{\infty}) = \infty$, evaluating the log a few more times will yield

$$\frac{A}{(f(\log(\widehat{\infty}))} = 0 \quad .$$

This shows that the expression is always less than $a \in \mathbb{R}_0^+$. Therefore, the elements of $\{\gamma_n\}$ form an unbroken line when $|\text{Im}(z)| \in \mathbb{R}_\infty$. This proves the theorem.

Remark 8.2.6 Note that $\{\gamma_n\}$ is not such that each element can be labeled with $n \in \mathbb{N}$ because the zeros become uncountably infinite in the neighborhood of infinity. Rather, $\{\gamma_n\}$ must be a sequence in the sense that it is an ordered set of mathematical objects, some of which are intervals. Also note that $\{\gamma_n\}$ is a proper sequence in the usual sense when we take $n \in \mathbb{N}_\infty$, as in Definition 6.2.3. The extended natural numbers are uncountably infinite.

Corollary 8.2.7 *The Riemann ζ function has zeros within the critical strip yet off the critical line.*

Proof. Proof follows from Axiom 8.2.3 and Main Theorem 8.2.5. If the imaginary parts of the zeros form an unbroken line in the neighborhood of infinity, then the zeros are not isolated. Since ζ is holomorphic on $\mathbb{C}^{\cup} \setminus Z_1$, it must have zeros everywhere on a disc of radius $r_0 \in \mathbb{R}_0$ of any of the zeros on the critical line. Some of these zeros, obviously, are within the critical strip yet not on the critical line.

Remark 8.2.8 The Riemann hypothesis is false.

§8.3 Non-trivial Zeros in the Neighborhood of Minus Infinity

The trivial zeros of the Riemann ζ function are the negative even integers $z = -2, -4, -6...$ [30]. In this section, we will prove that ζ has non-trivial zeros outside of the critical strip. The theorem of Hadamard and de la Vallée-Poussin [31, 32] is usually taken to rule out the existence of such zeros so here we will conjecture that the theorem fails in the neighborhood of infinity. Indeed, it follows from Corollary 8.2.7 that ζ has zeros on the line $\mathrm{Re}(z) = 1$ and this is something else which contradicts the theorem of Hadamard and de la Vallée-Poussin. We will conjecture that their result fails in the neighborhood of infinity, most likely due to something about quotients of the form $\mathbb{R}_0 \div \mathbb{R}^{\mathcal{X}}$ being surprising identical zeros.

Theorem 8.3.1 *The Riemann ζ function is equal to one for any $\mathrm{Re}(z) \in \mathbb{R}^{\mathcal{X}}$ such that $0 < \mathcal{X} \leq 1$.*

Proof. Observe that the Dirichlet form of ζ

$$\zeta(z) = \sum_{n\in\mathbb{N}} \frac{1}{n^z} \quad ,$$

takes $z_0 = \big(\aleph_{\mathcal{X}} + b\big) + iy$ as

$$\begin{aligned}
\zeta(z_0) &= \sum_{n=1} \frac{1}{n^{(\aleph_{\mathcal{X}}+b)+iy}} \\
&= \sum_{n=1} \frac{n^{-b} n^{-iy}}{n^{\aleph_{\mathcal{X}}}} \\
&= \sum_{n=1} \frac{n^{-b}}{\big(n^{\mathcal{X}}\big)^{\widehat{\infty}}} \Big(\cos(y \ln n) - i \sin(y \ln n) \Big) \\
&= 1 + \sum_{n=2} \frac{n^{-b}}{\infty} \Big(\cos(y \ln n) - i \sin(y \ln n) \Big) \\
&= 1 \quad .
\end{aligned}$$

Main Theorem 8.3.2 *The Riemann ζ function has non-trivial zeros z_0 such that $-\text{Re}(z_0) \in \mathbb{R}^{\mathcal{X}}$ for $0 < \mathcal{X} \leq 1$. In other words, ζ has non-trivial zeros in the neighborhood of minus real infinity.*

Proof. Definition 8.1.6 gives Riemann's functional form of ζ [16] as

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

Theorem 8.3.1 gives $\zeta(\aleph_{\mathcal{X}} + b) = 1$ when we set $y = 0$ so we will use Riemann's equation to prove the present theorem by computing $\zeta(z)$ at $z_0 = -(\aleph_{\mathcal{X}}+b)+1$. (This value for z_0 follows from $1 - z_0 = \aleph_{\mathcal{X}} + b$.) We have

$$\begin{aligned} \zeta\big[-(\aleph_{\mathcal{X}} + b) + 1\big] &= \lim_{z \to -(\aleph_{\mathcal{X}}+b)+1} \left(\frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \right) \lim_{z \to (\aleph_{\mathcal{X}}+b)} \Big(\Gamma(z)\zeta(z) \Big) \\ &= \lim_{z \to -(\aleph_{\mathcal{X}}+b)+1} \left(2 \sin\left(\frac{\pi z}{2}\right) \right) \lim_{z \to (\aleph_{\mathcal{X}}+b)} \Big((2\pi)^{-z}\Gamma(z)\zeta(z) \Big) \quad . \end{aligned} \tag{8.1}$$

For the limit involving Γ, we will compute the limit as a product of two limits. We separate terms as

$$\lim_{z \to (\aleph_{\mathcal{X}}+b)} \Big((2\pi)^{-z}\Gamma(z)\zeta(z) \Big) = \lim_{z \to (\aleph_{\mathcal{X}}+b)} \Big((2\pi)^{-z}\Gamma(z) \Big) \lim_{z \to (\aleph_{\mathcal{X}}+b)} \zeta(z) \quad .$$

From Theorem 8.3.1, we know the limit involving ζ is equal to one. For the remaining limit, we will insert the identity and again compute it as the product of two limits. If z approaches $(\aleph_{\mathcal{X}} + b)$ along the real axis, then it follows from Axiom 5.2.11 that

$$1 = \frac{z - (\aleph_{\mathcal{X}} + b)}{z - (\aleph_{\mathcal{X}} + b)} \quad .$$

Inserting the identity yields

$$\lim_{z \to (\aleph_{\mathcal{X}}+b)} \Big((2\pi)^{-z}\Gamma(z) \Big) = \lim_{z \to (\aleph_{\mathcal{X}}+b)} \left((2\pi)^{-z}\Gamma(z) \frac{z - (\aleph_{\mathcal{X}} + b)}{z - (\aleph_{\mathcal{X}} + b)} \right) \quad .$$

Let

$$\alpha = \Gamma(z)\Big(z - (\aleph_{\mathcal{X}} + b) \Big) \quad , \quad \text{and} \quad \beta = \frac{(2\pi)^{-z}}{z - (\aleph_{\mathcal{X}} + b)} \quad .$$

To get the limit of α into workable form, we will use the property $\Gamma(z) = z^{-1}\Gamma(z+1)$ to derive an expression for $\Gamma[z - (\aleph_{\mathcal{X}} + b) + 1]$. If we can write $\Gamma(z)$ in terms of $\Gamma[z - (\aleph_{\mathcal{X}} + b) + 1]$, then the limit as z approaches $(\aleph_{\mathcal{X}} + b)$ will be very easy to compute. Observe that

$$\Gamma\big[z - (\aleph_{\mathcal{X}} + b) + 1\big] = \Gamma\big[z - (\aleph_{\mathcal{X}} + b) + 2\big]\Big(z - (\aleph_{\mathcal{X}} + b) + 1 \Big)^{-1} \quad .$$

On the RHS, we see that Γ's argument is increased by one with respect to the Γ function that appears on the LHS. The purpose of inserting the identity

$$\frac{z-(\aleph_{\mathcal{X}}+b)}{z-(\aleph_{\mathcal{X}}+b)}=1 \quad ,$$

was precisely to exploit this self-referential identity of the Γ function which is most generally expressed as

$$\Gamma\big(z\big)=\Gamma\big(z+1\big)z^{-1} \quad .$$

By taking a limit of recursion, we will let z approach a number in the neighborhood of infinity. Then through the axiomatized addition of such numbers (Axiom 5.2.3), we will cast the argument of Γ into the neighborhood of the origin where Γ's properties are well-known. The limit is

$$\Gamma\big[z-(\aleph_{\mathcal{X}}+b)+1\big]=\Gamma(z)\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=1}^{n}\bigg(z-(\aleph_{\mathcal{X}}+b)+k\bigg)^{-1} \quad .$$

Moving the infinite product to the other side yields

$$\Gamma(z)=\Gamma\big[z-(\aleph_{\mathcal{X}}+b)+1\big]\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=1}^{n}\bigg(z-(\aleph_{\mathcal{X}}+b)+k\bigg) \quad .$$

We have let $\alpha=\Gamma(z)(z-(\aleph_{\mathcal{X}}+b))$ where the coefficient $z-(\aleph_{\mathcal{X}}+b)$ can be expressed as the $k=0$ term in the infinite product. It follows that

$$\alpha=\Gamma\big[z-(\aleph_{\mathcal{X}}+b)+1\big]\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=0}^{n}\bigg(z-(\aleph_{\mathcal{X}}+b)+k\bigg) \quad .$$

To evaluate the limit of $\alpha\beta$, we will take the limits of α and β separately. The limit of α is

$$\lim_{z\to(\aleph_{\mathcal{X}}+b)}\alpha=\Gamma\big[(\aleph_{\mathcal{X}}+b)-(\aleph_{\mathcal{X}}+b)+1\big]\times$$

$$\times\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=0}^{n}\bigg((\aleph_{\mathcal{X}}+b)-(\aleph_{\mathcal{X}}+b)+k\bigg) \quad .$$

Axiom 5.2.3 gives $(\aleph_{\mathcal{X}}+b)-(\aleph_{\mathcal{X}}+b)=0$ so

$$\lim_{z\to(\aleph_{\mathcal{X}}+b)}A=\Gamma(1)\lim_{n\to(\aleph_{\mathcal{X}}+b)}\prod_{k=0}^{n}k=0 \quad .$$

Direct evaluation of the $z\to(\aleph_{\mathcal{X}}+b)$ limit of $\beta=(2\pi)^{-z}(z-(\aleph_{\mathcal{X}}+b))^{-1}$ gives $\frac{0}{0}$ so we need to use L'Hôpital's rule. Evaluation yields

$$\lim_{z\to(\aleph_{\mathcal{X}}+b)}\beta\stackrel{*}{=}\lim_{z\to(\aleph_{\mathcal{X}}+b)}\left(\frac{\dfrac{d}{dz}(2\pi)^{-z}}{\dfrac{d}{dz}\bigg(z-(\aleph_{\mathcal{X}}+b)\bigg)}\right)$$

$$
\begin{aligned}
&= \lim_{z\to(\aleph_{\mathcal{X}}+b)} \frac{d}{dz} e^{-z\ln(2\pi)} \\
&= -\ln(2\pi)\ e^{-(\aleph_{\mathcal{X}}+b)\ln(2\pi)} \\
&= -\ln(2\pi)\ \frac{e^{-b\ln(2\pi)}}{\left(e^{\mathcal{X}\ln(2\pi)}\right)^{\widehat{\infty}}} \\
&= -\ln(2\pi)\frac{e^{-b\ln(2\pi)}}{\infty} \\
&= 0 \quad .
\end{aligned}
$$

By Axiom 5.1.1 giving $\frac{1}{\infty} = 0$, we find that the limit of $\alpha\beta$ is 0. It follows from Equation (8.1) that

$$
\zeta\big[-(\aleph_{\mathcal{X}}+b)+1\big] = \lim_{z\to-(\aleph_{\mathcal{X}}+b)+1} 2\sin\left(\frac{\pi z}{2}\right)\cdot 0 = 0 \quad .
$$

Example 8.3.3 To demonstrate that Riemann's functional form of ζ is robust, we should check for consistency by reversing the sign of z and $1-z$ to show that there is no contradiction. What this means is that we have computed a value in the left complex half-plane using a known value in the right complex half-plane (Theorem 8.3.1), and now we will use the newly found value on the left (Main Theorem 8.3.2) to see what it says about the value on the right. We have

$$
\Gamma(-\aleph_{\mathcal{X}}+1) = \frac{1}{-\aleph_{\mathcal{X}}+1}\prod_{n=1}^{\infty}\left(1-\aleph_{\left(\frac{\mathcal{X}}{n}\right)}+\frac{1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{-\aleph_{\mathcal{X}}+1} = 0 \quad ,
$$

and we have shown in Main Theorem 8.3.2 that $\zeta(-\aleph_{\mathcal{X}}+1) = 0$. Using Riemann's formula

$$
\zeta(z) = \frac{(2\pi)^z}{\pi}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z) \quad ,
$$

to derive the relationship between $\zeta(z)$ and $\zeta(1-z)$, we will compute $\zeta(\aleph_{\mathcal{X}})$. Evaluation yields

$$
\zeta(\aleph_{\mathcal{X}}) = \left[2(2\pi)^{\aleph_{\mathcal{X}}-1}\sin\left(\frac{\pi\aleph_{\mathcal{X}}}{2}\right)\right]\Gamma(-\aleph_{\mathcal{X}}+1)\zeta(-\aleph_{\mathcal{X}}+1) = [\infty](0)(0) \quad .
$$

This equation is undefined and we cannot obtain a contradiction. This example has demonstrated the robust character of Riemann's functional equation in the neighborhood of infinity. It has also demonstrated why we must take $x^{\widehat{\infty}} = \infty$ for $x > 1$ (Axiom 5.1.7.). If this expression was said to be equal to algebraic infinity, as in $x^{\widehat{\infty}} = \widehat{\infty}$, then Axiom 5.1.4 giving $\widehat{\infty}\cdot 0 = 0$ would produce a contradiction in Riemann's functional equation under the reversal of z and $1-z$.

Remark 8.3.4 If one requires that Riemann's functional equation can never be undefined, meaning that it is not sufficient for the equation to simply determine ζ on left complex half-plane from its behavior on the right, but that it must determine one equally from the other, then we must introduce a convention such that $\infty \cdot 0 = 1$. With this definition, the derivation followed in Example 8.3.3 would confirm Theorem 8.3.1 giving $\zeta(\aleph_{\mathcal{X}}) = 1$. For many reasons, the product $0 \cdot \infty$ is taken as undefined and yet there are certain realms of mathematics in which it is given the definition $0 \cdot \infty = 1$. Therefore, one would explore whether or not a scheme of transfinite numbers as the 1D longitudinal analytic continuation of $\mathbb{R}$ onto $\mathbb{T}$ via the order relation $|\widehat{\infty}| < \infty$ might allow for $0 \cdot \infty = 1$. For the purposes of the Riemann hypothesis, however, it is sufficient that the functional equation determines ζ on the left complex half-plane without invoking a contradiction.

Remark 8.3.5 Patterson writes the following in reference [18].

> "There is a second representation of ζ due to Euler in 1749 which [*sic*] is the reason for the significance of the zeta-function. This is
>
> $$\zeta(s) = \prod_{p\,\in\,\text{primes}} \left(1 - p^{-s}\right)^{-1} \quad ,$$
>
> where the product is taken over all prime numbers p. This is called the Euler Product representation of the zeta-function and gives analytic expression to the fundamental theorem of arithmetic."

The fundamental theorem of arithmetic is given in *The Elements* [1] as Book 7, Propositions 30, 31, and 32. A modern statement of the fundamental theorem of arithmetic is that every natural number greater than one is a prime number or it is a product of prime numbers. The ultimate goal of all of number theory being concerned with the distribution of the prime numbers, now we will demonstrate as a corollary result that the Euler product form of ζ [18,33] shares at least some zeros with the the Riemann ζ function in the left complex half-plane where the absolute convergence of the Euler product to the Riemann ζ function is not historically proven.

Corollary 8.3.6 *The Euler product from of ζ has non-trivial zeros with negative real parts in $\mathbb{R}_\infty$.*

Proof. Consider a number $z_0 \in \mathbb{C}$ such that

$$z_0 = -(\aleph_{\mathcal{X}} + b) + iy \quad , \quad \text{where} \qquad b, y \in \mathbb{R}_0 \quad .$$

Observe that the Euler product form of ζ [33] takes z_0 as

$$\zeta(z_0) = \prod_p \frac{1}{1 - p^{(\aleph_{\mathcal{X}} + b) - iy}}$$

$$= \left(\frac{1}{1 - P^{(\mathcal{X}\aleph_1 + b) - iy}} \right) \prod_{p \neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}} + b) - iy}}$$

$$= \left(\frac{1}{1 - P^b \left(P^{\mathcal{X}}\right)^{\widehat{\infty}} \left[\cos(y \ln P) - i \sin(y \ln P)\right]} \right) \prod_{p \neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}} + b) - iy}} \quad .$$

Let $y \ln P = 2n\pi$ for some prime P and $(n+1) \in \mathbb{N}$. Then

$$\zeta(z_0) = \frac{1}{\infty} \prod_{p \neq P} \frac{1}{1 - p^{(\aleph_{\mathcal{X}} + b) - iy}} = 0 \quad .$$

Conjecture 8.3.7 The theorem of Hadamard and de la Vallée-Poussin [31,32] showing that ζ never vanishes on the line $\mathrm{Re}(z) = 1$ should fail along the portions of that line lying in the neighborhood of infinity. Likewise, the result proving that ζ cannot have any zeros beyond the critical strip other than the negative even integers must fail in the neighborhood of infinity.

§A Developing Mathematical Systems Historically

Because this treatise so concisely follows a *very* long trail of preexisting philosophical pursuits in mathematics, we present here as an appendix a concise summary of some of the important questions which motivated the modernist approach to complementing Euclid as the foundation of real analysis. In the article *The Real Numbers: From Stevin to Hilbert*, O'Connor and Robertson write the following [34].

> "By the time Stevin proposed the use of decimal fractions in 1585, the concept of a real number had developed little from that of Euclid's *Elements*. Details of the earlier contributions are examined in some detail in our article *The real numbers: from Pythagoras to Stevin*."

This appendix summarizes two articles by O'Connor and Robertson which outline the history of what are called the real numbers today [34,35]. This appendix essentializes the trail of facts supporting the present axiom-constructive fractional distance approach to the real number system. Setting the stage for the theme, O'Connor and Robertson write the following.

> "By the beginning of the 20th century then, the concept of a real number had moved away completely from the concept of a number which had existed from the most ancient times to the beginning of the 19th century, namely its connection with measurement and quantity."

O'Connor and Robertson cite Wallis as writing the following.

> "[S]uch proportion is not to be expressed in the commonly received ways of notation."

Wallis makes a wholehearted declaration of the mathematical matter contended by fractional distance. Sometimes it is necessary to introduce new notations such as $\aleph_{\mathcal{X}}$, $\widehat{\infty}$, and $\mathbb{F}^{\mathcal{X}}$. Therefore, should it be claimed that one may not simply declare a thing such as $\aleph_{\mathcal{X}}$, Wallis is cited as evidence that one may and that, at times, one must. Further emphasizing the importance of the influx of new notations into contemporary mathematics, O'Connor and Robertson write the following.

> "A major advance was made by Stevin in 1585 in *La Thiende* when he introduced decimal fractions. One has to understand here that in fact it was in a sense fortuitous that his invention led to a much deeper understanding of numbers for he certainly did not introduce the notation with that in mind. Only finite decimals were allowed, so with his notation only certain rationals [*were*] to be represented exactly. Other rationals could be represented approximately and

> Stevin saw the system as a means to calculate with approximate rational values. His notation was to be taken up by Clavius and Napier but others resisted using it since they saw it as a backwards step to adopt a system which could not even represent $\frac{1}{3}$ exactly."

Still yet further emphasizing the rightful place of new notation in mathematics, O'Connor and Robertson write the following.

> "Strictly speaking, only that which is logically impossible (i.e.: which contradicts itself) counts as impossible for the mathematician."

All progress in mathematics, therefore, must be predicated from time to time upon the introduction of new notations such as $\aleph_{\mathcal{X}}$ and $\widehat{\infty}$.

Now we have shown the aesthetic likeness of the present course to the previous course. Stevin introduced decimal fractions and now we have introduced infinity hat. Leibniz gave us the integral symbol and now there exists a real number $\aleph_{0.5}$ (which was already known as long as ago Euler who wrote $\frac{i}{2}$ [7].) Now we will emphasize that the course in question has always been the means by which to unify algebra and geometry. O'Connor and Robertson write the following.

> "Similarly Cantor realized that if ***he wants the line to represent the real numbers*** [*emphasis added*] then he has to introduce an axiom to recover the connection between the way real numbers are now being defined and the old concept of measurement."

O'Connor and Robertson specifically identify Cantor's motivations [34] as the same given here. How can we best preserve the geometric notion of an infinite line in the algebraic arena? If one supposes that "infinity is not allowed," and lets that be the end of the inquiry into the preservation of the notion of infinite geometric extent, then it is unlikely that the resulting mathematical system will make sufficient provisions for that fundamental notion. Indeed, the entire theme of this work has been to *modify* existing mathematical systems so as to better accommodate the notion of infinite geometric extent. Cantor himself wrote the following.

> "[*O*]ne may add an axiom which simply says that every numerical quantity also has a determined point on the straight line whose coordinate is equal to that quantity."

In the present treatise, we have extended Cantor to separately consider the "determined" geometric point from the numbers in the algebraic representation of that point. Indeed, this is the main distinction between our own approach and Cantor's approach. This issue fairly well represents the issue cited in Remark 3.1.20 as the source of "much pathology" in modern analysis: Cantor's presumption of a one-to-one correspondence between numbers and points is a fair proxy for one's choice to distinguish algebraic FDFs of the first and second kinds. Cantor's implied concept of fractional distance seems to favor $\mathcal{D}^{\dagger}_{AB} = \mathcal{D}''_{AB}$ whereas we have demonstrated the philosophical superiority of $\mathcal{D}^{\dagger}_{AB} = \mathcal{D}'_{AB}$. We can glean from Cantor's words that he likely associated only one number with each point but we have shown that this is only best when the line segment is of finite length. If the determined point is in an infinitely long line segment such as $X \in \mathbf{AB}$, then we have proven that the determined point does not have one uniquely associated real number.

In this treatise, we have restated the ancient Archimedes property with English and Latin mathematical symbols (Definition 6.3.9.) We have also given a modern restatement of the Archimedes property as the Archimedes property of 1D transfinitely continued real numbers (Axiom 6.3.12.) Similarly, Hilbert gave his own modernized restatement of that property when giving his geometry axioms [4]. O'Connor and Robertson write the following.

> "[*Hilbert's statement of the Archimedes property was*] that given positive numbers a and b then it is possible to add a to itself a finite number of times so that the sum exceeds b."

What Hilbert wrote specifically was the following.

> "If AB and CD are any segments then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B."

Hilbert's original reliance on the AB notation to give a statement of the Archimedes property for a Euclidean line segment very strongly highlights the historical similitude of the present approach to a modernizing algebraic capstone on Euclidean geometry. Hilbert's axioms of geometry applied to Dedekind cuts give us the field axioms, more or less, so it is remarkable that we were likewise called, while working to the same ends as Hilbert, to give a restatement of what Euclid meant when he said he had it on good authority that Archimedes had heard from Eudoxus that such and such was the real Archimedes property of real numbers. In the case of Hilbert's statement of the Archimedes property, we see that Hilbert gave a finite multiplier but did not explicitly require $n \in \mathbb{N}$. The extended natural numbers $n \in \mathbb{N}_\infty$ provide the multipliers needed to preserve Hilbert's statement of the property in the fractional distance approach to real analysis.

Regarding the very ancient history, O'Connor and Robertson write the following.

> "It seems clear that Pythagoras would have thought of $1, 2, 3, 4 \ldots$ (the natural numbers in the terminology of today) in a geometrical way, not as lengths of a line as we do, but rather in the form of discrete points. Addition, subtraction, and multiplication of integers are natural concepts with this type of representation but there seems to have been no notion of division."

Even as long as ago as Pythagoras, the open question of the separation of algebraic numbers from geometric magnitudes was already one of import. Also, we have a distinct and apt likeness here with the possibility that we might give the regularly-spaced, disconnected immeasurable numbers $\mathcal{F}_{\mathcal{X}} \in \mathbb{F}$ an arithmetic axiom such that they are the real numbers on an infinitely bigger copy of the real number line.

In the present treatise, like Hilbert very recently, we have sought to build a hybrid constructive framework for mathematical analysis which maximizes the synergy between algebra and geometry. O'Connor and Robertson write the following.

> "[*I*]t should be mentioned at this stage that the Egyptians and the Babylonians had a different notion of a number to that of the ancient Greeks. The Babylonians looked at reciprocals and also at approximations to irrational numbers, such as $\sqrt{2}$, long before Greek mathematicians considered approximations. The Egyptians also looked at approximating irrational numbers.
>
> "Let us now look at [*sic*] Euclid's *Elements*. This is an important stage since it would remain the state of play for nearly the next 2000 years. In Book 5 Euclid considers magnitudes and the theory of proportion of magnitudes. It is probable (and claimed in a later version of *The Elements*) that this was the work of Eudoxus. Usually when Euclid wants to illustrate a theorem about magnitudes he gives a diagram representing the magnitude by a line segment. However magnitude is an abstract concept to Euclid and applies to lines, surfaces and solids. Also, more generally, Euclid also knows that his theory applies to time and angles.
>
> "Given that Euclid is famous for an axiomatic approach to mathematics, one might expect him to begin with a definition of magnitude and state some unproved axioms. However he leaves the concept of magnitude undefined and his first two definitions refer to the part of a magnitude and a multiple of a magnitude."

O'Connor and Robertson proceed to break down Euclid's Book 5 as we have when examining the Archimedes property in Section 6.3. Therefore, we will list the properties and comments again in expanded form. We consolidate the supplemental comments to Euclid's original text with Fitzpatrick's labeled (RF), our own comments labeled (JT), and the comments of O'Connor and Robertson labeled (OR).

Book 5, Definition 1 A magnitude is a part of a(nother) magnitude, the lesser of the greater, when it measures the greater.

> (RF) In other words, α is said to be a part of β if $\beta = m\alpha$.
>
> (JT) The first definition makes it obvious that the multiplier is not meant to be a natural number. If the magnitude of ten units of

> geometric length is to be greater than one of nine, then there must exist non-integer multipliers.
>
> (OR) Again the term "measures" here is undefined but clearly Euclid means that (in modern symbols) the smaller magnitude x is a part of the greater magnitude y if $nx = y$ for some natural numbers $n > 1$.

When O'Connor and Robertson write $n \in \mathbb{N}$, they do not take into consideration numbers having non-integer quotients, *e.g.*: $10:9$, or else they are only giving a subcase of what is meant in the original context. Using the natural numbers to demonstrate the property makes sense if one takes the auxiliary axiom that there are no real numbers greater than every natural number. In that case, the Archimedes property is irreducibly represented in the natural number statement of the multiplier. The supposition that every real number is in the neighborhood of the origin was a normal axiom at the time of the publication of References [34,35]. The main difference between the present approach and the historical approaches to merging geometry and algebra is that we have not tried to squeeze the notion of geometric infinity into the algebraic sector. In the present conventions, $\widehat{\infty}$ is such that the algebraic structure is totally subordinate to the geometric structure. The primary theme of the past few centuries has been one of attempting to subordinate geometry to algebra but we have eschewed that effort in taking the fractional distance tack.

Many historical approaches have assumed some algebraic axioms and then tried to fit everything inside those axioms by ignoring geometric infinity and making a rule that one must never mention it. Note the equal weighting of the gravity of the matters in the choice to suppose one of the two following axioms.

Axiom A.1 There exists a non-empty set of real numbers greater than any natural number.

Axiom A.2 There does not exist any real number greater than every natural number.

An assigned superiority in the algebraic sector might make Axiom A.2 the more attractive axiom because it allows everything to be written with the field axioms. By assigning the superior quality as the historical geometric conception of numbers, we are drawn to Axiom A.1 as the preferable axiom. Additionally, we have proven multiply that Axiom A.2 causes undesirable contradictions with the geometric notion of fractional distance. Even when algebraic considerations are chosen as superior to geometric ones, the superior axiom must not contradict its inferior complement. The neighborhood of infinity does exist; fractional distance requires it. The question is only whether we should adopt an algebraic convention which reflects the geometric reality.

Book 5, Definition 2 And the greater is a multiple of the lesser whenever it is measured by the lesser.

> (JT) This definition makes it explicitly clear that the manner in which one magnitude may measure another is such that, for example, nine can measure ten by $10:9$.
>
> (OR) Then comes the definition of ratio.

Book 5, Definition 3 A ratio is a certain type of condition with respect to size of two magnitudes of the same kind.

> (RF) In modern notation, the ratio of two magnitudes, α and β, is denoted $\alpha : \beta$.
>
> (JT) This definition tells us that $\mathbb{R}$ is equipped with $\leq$ relation. The specification of two magnitudes of the same kind tells us, essentially, that Euclid does not want his reader to compare lengths with areas, volumes, angles, hypervolumes, *etc.* Likewise, once we have conjured $\mathbb{R}_{\mathbb{T}}$ from an interpretation of $\mathcal{F}(n) \in \mathbb{R}$ as $n \in \mathbb{N}_{\mathbb{T}}$, we must be careful to distinguish the underlying magnitudes as different kinds.
>
> (OR) This is an exceptionally vague definition of ratio which basically fails to define it at all. [*Euclid*] then defines when magnitudes have a ratio, which according to the definition is when there is a multiple (by a natural number) of the first which exceeds the second and a multiple of the second which exceeds the first.

Book 5, Definition 4 (Those) magnitudes are said to have a ratio with respect to one another which, being multiplied, are capable of exceeding one another.

> (RF) In other words, α has a ratio with respect to β if $m\alpha > \beta$ and $n\beta > \alpha$, for some m and n.
>
> (JT) The Archimedes property of real numbers requires that for every real number, there is a greater real number. In other words and in a general way, there is no largest real number because $\aleph_1 \notin \mathbb{R}$. Usually it is said that a smallest real number is also precluded by the inverse of the unbounded large number. Surprisingly, the usual topology requirement of the fundamental axiom of algebraic construction (Axiom 7.6.8 and/or Axiom 2.1.7) seems to indicate that a smallest real number must exist (Proposition 7.2.11.) This is the $\mathcal{X}_{\text{min}}$ value of $\aleph(2) = \aleph_{\mathcal{X}_{\text{min}}}$. The issue of a smallest positive real number has been a historically vexing contention in the intuitive sense. This is demonstrated in the following manner. If every interior point in a connected interval $(-1, 1) \subset \mathbb{R}$ is left- and right-adjacent

to another point, meaning the interval is not disconnected, then writing

$$(-1,1) = (-1,0] \cup (0,1) \quad ,$$

suggests, in an intuitive way at least, that zero must be left-adjacent to the smallest positive real number. However, the protocols of mathematics override intuition and it is said that zero is not left-adjacent to any element of $(0,1)$ because every element of $(0,1)$ has a δ-neighborhood lying totally within $(0,1)$. So, if some way is found to claw a least positive real number from the precepts of fractional distance, then the concept of no greatest real number would also have to be done away with due to the invariance of **AB** under permutations of the labels of its endpoints. Infinity minus the least positive real number would be the greatest real number.

(OR) The Archimedean axiom stated that given positive numbers a and b then it is possible to add a to itself a finite number of times so that the sum exceed b.

Book 5, Definition 5 Magnitudes are said to be in the same ratio, the first to the second, and the third to the fourth, when equal multiples of the first and third both exceed, are both equal to, or are both less than, equal multiples of the second and fourth, respectively, being taken in corresponding order, according to any kind of multiplication whatever.

(RF) In other words, $\alpha : \beta :: \gamma : \delta$ if and only if $m\alpha > n\beta$ whenever $m\gamma > n\delta$, $m\alpha = n\beta$ whenever $m\gamma = n\delta$, and $m\alpha < n\beta$ whenever $m\gamma < n\delta$, for all m and n. This definition is the kernel of Eudoxus' theory of proportion, and is valid even if α, β, *etc.*, are irrational.

(JT) This definition gives the trichotomy of the $\leq$ relation. Also note that the ratio of ratios is like the ratio of two fractional distances.

(OR) Then comes the vital definition of when two magnitudes are in the same ratio as a second pair of magnitudes. As it is quite hard to understand in Euclid's language, let us translate it into modern notation. It says that $a : b = c : d$ if given any natural numbers n and m we have

$$\begin{aligned} na > mb &\quad \text{if and only if} \quad nc > md \\ na = mb &\quad \text{if and only if} \quad nc = md \\ na < mb &\quad \text{if and only if} \quad nc < md \quad . \end{aligned}$$

Euclid then goes on to prove theorems which look to a modern mathematician as if magnitudes are vectors, integers are scalars, and he is proving the vector space axioms.

The main hurdle in the vector space conception of $\mathbb{R}$ is that the product of two vectors is a scalar but the product of two real numbers is another real number. Even in the transfinite continuation beyond algebraic infinity, and even when the product of two things in the line always remains within the geometrically infinite line as if it were a vector space, the problem remains that the product of two 1D transfinitely continued extended real numbers will be another 1D transfinitely continued extended real number. There is no distinguishing a vector from a scalar. However, one easily imagines $\widehat{\infty}$ as an anchor point for 1D vectors $x \in \mathbb{R}$ different than the anchor point at the origin. Vectors anchored in the neighborhood of the origin look like $\hat{0} + \vec{b}$ and those anchored in the neighborhood of positive infinity look like $\widehat{\infty} - \vec{b}$. The 1D vector space picture is very becoming the notion of a 1D geometric space but the lack of distinction among vectors and scalars forbids any approach to the commonly stated modern vector space axioms.

§B Toward Mathematical Physics

To the extent that this present fractional distance work in pure mathematics grew from, and was motivated by, a preexisting research program in theoretical physics, there are some things should be pointed out. The first salient point regards the finite or infinite interaction ranges of the fundamental forces. Gravity and electromagnetism are said to have infinite ranges because the relevant classical forces between massive or charged particles go as $\frac{1}{r^2}$. When all finite numbers are assumed to be less than some natural number, it is a direct consequence that such inverse squared force laws can never go to zero for any $r \in \mathbb{R}$. The arithmetic axioms, however, allow these forces to go to zero for any finite separation in the neighborhood of infinity. Indeed, the entire initial inquiry into infinity which eventually resulted in the fruits presented here was the following question: how might we have two physical objects (where physicality requires that they are separated in spacetime by less than infinite spacetime interval) whose mutual gravitational interaction is precisely zero? Now $r \in \mathbb{R} \setminus \mathbb{R}_0$ provides exactly the requisite finite scale. As an example for how the neighborhood of infinity might be worked into the progenitive cosmological scenario which spurred this research (the modified cosmological model (MCM) [36, 37]), we could set the scale of $\mathcal{F}_0$ as the $\sim$13.7Gcy radius of the observable universe. Nothing beyond the cosmic microwave background (CMB) at that distance can be observed yet it might be helpful for the development of new theories if we could set the interactions between the local frame and the occulted region beyond the CMB to an identical zero rather than the almost zero which always has a big impact on quantum considerations over cosmological timescales.

Another good use for the neighborhood of infinity is the mathematical description of wavepackets. In the efforts of physicists to describe the wave-particle duality of quantum particles, the quantum states are formally rendered as enveloped wave-packets whose tails extend to infinity. In many situations, the non-vanishing tails of these wavepackets can be ignored. For instance, if the probability that an electron will be observed in a lab during one moment and then observed 40cy from the Earth in the next moment is on the order of one in 10^{-50}, then we may treat this as zero probability and proceed accordingly without shooting our theories in the foot. However, in the regime of ultra-fast quantum optics, the tails of mathematical wave-packets describing picosecond laser pulses generate notoriously vexing discrepancies with what is observed in the lab. Having developed the neighborhood of infinity, one might develop a language for wave-packets whose tails go to zero on the scale of $\mathcal{F}_0$ while the universe itself could be said to have a characteristic scale on the order of $\aleph_1$. Any number of such schemes could be developed.

Throughout the research in cosmology which led to the present fractional distance analysis, we have developed a requirement for some intermediate scale between the scale of natural numbers and the scale of infinity. Specifically, we have used the concept of "odd and even levels of aleph" to describe the behavior that is required for the underlying cosmological model [36–39]. In the language of Robinson's hyperreal analysis, we have limiteds and unlimiteds but the model which spurred the present analysis generated a requirement for some intermediate scale: the odd level of aleph. In that language, we say that infinity is two levels of aleph higher than the level of aleph upon which resides the origin of the abstract inertial lab frame. Robinson's canonical method for mixing limiteds and unlimiteds only provides tool for the even levels of aleph. In the present work, we have generated the "super-finite" $\mathcal{F}_0$ scale which can serve as the odd level of aleph between Robinson's limiteds and unlimiteds. See Reference [40] for a brief treatment of the Riemann hypothesis in terms of the odd and even levels of aleph.

In the course of developing the MCM, we have shown that the structure of the standard model of particle physics arises as the elementary structure of the unit cell of the model's lattice cosmology [38]. Now, in the present work, we have found yet more of the fundamental quantum numbers in the underlying analytical structure. Quantum mechanics has the curiously measurable half-inter spin quantization of fermions, and now we have demonstrated a half-integer interval of spacing inherent between the origins of the successively transfinite scaled copies of $\mathbb{R}$ (Theorem 7.5.25.) Furthermore, the number $\mathcal{F}(1)$ lies one third of the way down the interval $[0, \mathcal{F}(2)]$ so we also have the asymmetric fractional charges of the quarks: $-\frac{1}{3}e$ for three of them and $\frac{2}{3}e$ for the other three.

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