On the Riemann Zeta Function

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We discuss the Riemann zeta function, the topology of its domain, and make an argument against the Riemann hypothesis. While making the argument in the classical formalism, we discuss the material as it relates to the theory of infinite complexity (TOIC). We extend Riemann's own (planar) analytic continuation $\mathbb{R}\to\mathbb{C}$ into (bulk) hypercomplexity with $\mathbb{C}\to{}^\star\mathbb{C}$. We propose a solution to the Banach–Tarski paradox.

Consider the analytic continuation of the Dirichlet series

$$\mathcal{D}(z)=\sum_{N=1}^{\infty}\frac{1}{N^z}\quad,\quad\text{with}\quad z>1,\ z\in\mathbb{R}\quad, \tag{1}$$

onto complex numbers $\mathbb{C}$ via the Riemann zeta function

$$\zeta(\mathcal{Z})=\left(\frac{\displaystyle\int_0^\infty\frac{x^{(\mathcal{Z}-1)}}{e^x-1}\,dx}{\displaystyle\int_0^\infty\frac{x^{(\mathcal{Z}-1)}}{e^x}\,dx}\right),\quad\text{with}\quad\text{Re}(\mathcal{Z})>1\quad. \tag{2}$$

We say ζ is interesting, among other reasons, because we can't be sure if it is the correct analytical form of the continuation for all $z\to\mathcal{Z}$. If $\zeta(\mathcal{Z})$ was well understood, and we could be sure that equation (2) is the correct form of the analytic continuation, then we would likely also know if it has any non-trivial zeros off the critical line.

The Riemann hypothesis says that all of the non-trivial zeros of $\zeta(\mathcal{Z})$ are such that if $\mathcal{Z}_n$ is a non-trivial zero then

$$\text{Re}(\mathcal{Z}_n)=\frac{1}{2}\quad. \tag{3}$$

Hardy proved in 1914 that there are an infinite number of these non-trivial zeros

$$\mathcal{Z}_n=\frac{1}{2}+i\gamma\quad,\qquad\text{with}\qquad\gamma\in\mathbb{R}\quad, \tag{4}$$

and, in this paper, we will make an argument that there should also exist at least one nontrivial zero such that

$$\zeta(\bar{\mathcal{Z}}_{\bar{n}})=0\quad,\qquad\text{with}\qquad\text{Re}(\bar{\mathcal{Z}}_{\bar{n}})\neq\frac{1}{2}\quad. \tag{5}$$

In our argument, we will take as a lemma that Hardy's proof of the existence of an infinite number of zeros on the critical line can be reduced to a symmetry argument that says an infinite number of zeros are on the critical line because it is in the center of the critical strip. If the argument cannot be so paraphrased then what follows will be of less relevance than hoped.

The domain of $\zeta(\mathcal{Z})$, $\mathbb{C}^Z$, is different than the extended complex plane $\hat{\mathbb{C}}$ which is defined as

$$\hat{\mathbb{C}}\quad\equiv\quad\mathbb{C}\cup\{\hat{\infty}_2\}\quad. \tag{6}$$

$\hat{\mathbb{C}}$ is also the 2-sphere $\mathbb{S}^2$ up to a complex phase factor so we have another definition

$$\hat{\mathbb{C}}\quad\equiv\quad\mathbb{S}^2\cup\{\hat{\imath}\}\quad. \tag{7}$$

What we will call the Riemann sphere shall be defined as

$$\mathbb{S}^R\quad\equiv\quad\hat{\mathbb{C}}\ /\ \{\theta=\pi\}\quad, \tag{8}$$

where θ is the zenith angle on $\mathbb{S}^2$. The domain of $\zeta(\mathcal{Z})$ is shown in figure 1 and its topology is

$$\mathbb{C}^Z\quad\equiv\quad\hat{\mathbb{C}}\ /\ \{\theta=\pi,\mathcal{Z}=1\}\quad. \tag{9}$$

The two null points $\theta\neq\pi$ and $\mathcal{Z}\neq1$ built into its definition show two distinct types of singularity: one is a polar coordinate singularity, and one is a singularity in the domain of analyticity of $\zeta(\mathcal{Z})$. We will use concepts of hypercomplexity to describe how these two types of singularities are different and can be used to make a good case for at least one novel, new non-trivial zero.

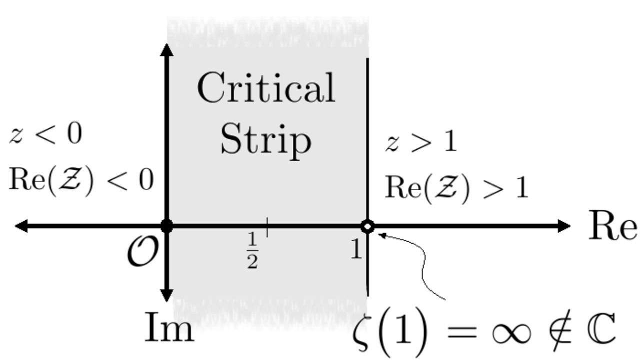

FIG. 1. This figure shows the features of the complex plane that are most relevant to the Riemann zeta function.

The analyticity of the Riemann zeta function means that its Taylor series expansion around every $\mathcal{Z}_0 \in \mathbb{C}^Z$ has to converge to ζ so $\zeta(1) = \infty$ means that ζ is not analytic at $\mathcal{Z}=1$. The point $\theta=\pi$ represents a defect in the domain of ζ but $\mathcal{Z}=1$ is a defect in its range. The overall topology of $\mathbb{C}^Z$ is the 2-sphere with one null polar point and another one due to

$$\zeta(1) \quad \equiv \quad \mathcal{D}(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... \;= \infty \quad . \tag{10}$$

Therefore, we see there are two points on the 2-sphere not in the domain of ζ: $\theta = \pi$ and $\mathcal{Z} = 1$. A 0-sphere is just two points so $\mathbb{C}^Z$ can be completely specified in terms of spheres as

$$\mathbb{C}^Z \quad \equiv \quad \mathbb{S}^2 \;/\; \mathbb{S}^0 \quad . \tag{11}$$

Since $\mathbb{S}^R \equiv \mathbb{C}$, the object $\hat{\infty}_2$ in figure 2 must act on $\mathbb{S}^R$ and $\mathbb{C}$ in the same fashion: it completes the topology with real and imaginary infinity. Despite adding only these two values, there are any number of directions that the $\hat{\infty}_2$ vector can point. To see this, consider the retracement of a planar spiral by the ray shown in figure 3. As the spiral approaches the open boundary of the non-extended complex plane, in which direction does the ray point? What is the direction of $\hat{\infty}_2$ at the boundary $r = \infty$ that is included with the extended complex plane? Shortly, we will show that the possible directions for $\hat{\infty}_2$ can be more complicated than the four planar boundaries $\{\pm\infty, \pm i\infty\}$ shown in figure 2. We will describe two new orientations for the hypercomplex version of $\hat{\infty}_2$ which we will call $\hat{\infty}_4$. The two new directions are concepts from the transfinite analysis of hyperreal numbers $^*\mathbb{R}$. These two added directions will be "beyond infinity" and "inside zero." The polar singularity X in figures 2 and 3 can be approached in $\hat{\mathbb{C}}$ from the $\hat{\imath}$ and $\hat{1}$ directions, but now we need to add details so that the null point can be approached from the big upper sphere or the small inner sphere, as in figure 4. We will do this with the hypercomplex number system $^\star\mathbb{C}$ which is the extension of the hyperreals $^*\mathbb{R}$ into complex numbers.

Nota bene, $\mathbb{C}^Z$ is not quite $\hat{\mathbb{C}}$. If the domain of ζ was $\hat{\mathbb{C}}$ then we would expect to be able to calculate $\zeta(\infty)$ but we have no such expectation. $\zeta(\infty)$ is neither well-defined nor an object we will consider until later. Per definitions (8) and (9), $\mathbb{C}^Z$ has two holes in it but $\mathbb{S}^R$ only has one. The domain $\mathbb{C}^Z$ of $\zeta(\mathcal{Z})$ will, therefore, due to equation (11), have properties that are universal under the sphere theorem in a way that cannot be accommodated by the topology of the Riemann sphere $\mathbb{S}^R$. (In the present context "the sphere theorem" can be understood as "partial differential equations.") $\mathbb{C}^Z$ has spherical topological symmetry but, perhaps, when one uses an analysis whose topology is the same for the analytic and polar singularities, *i.e.*: a nondescript topological pinprick, that causes a reliance on the $\mathbb{S}^R$ topology which causes a breakdown

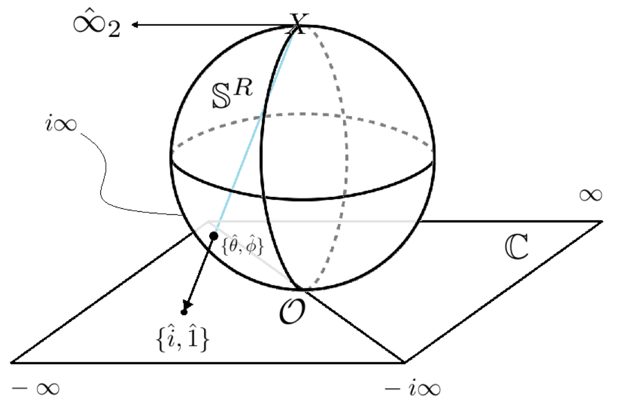

FIG. 2. The figure shows the stereographic projection of the complex plane onto a curved surface. To preserve the null topological component at $\theta = \pi$, which is a critical defining element of both $\mathbb{C}^Z$ and $\mathbb{S}^R$, we will mark it with an X.

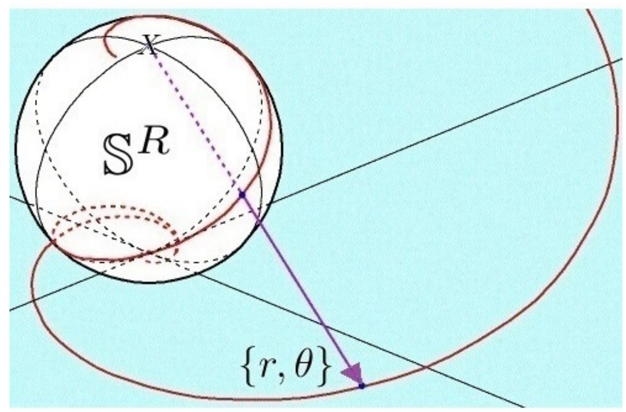

FIG. 3. By tracing the ray all the way out to $r = \infty$, this figure demonstrates that there is no unique direction for $\hat{\infty}_2$.

in what would otherwise be a useful symmetry in the domain of $\zeta(\mathcal{Z})$: that shown in figure 5. Said another way, perhaps when one treats analytical singularities as having identical topological properties to polar coordinate singularities, one inadvertently relies on $\mathbb{S}^R$ and its one type of null point which leads to a breakdown in analytical rigor resulting in an unanswered question about ζ's zeros.

What the sphere theorem shows is that $\mathbb{S}^N$, for any $N > 1$, can undergo a smooth deformation that swaps the sphere's interior region with its exterior region. Therefore, whatever we inscribe on the smallest sphere of figure 4 can undergo topological inversion to become the big exterior sphere, as in figure 5. In this paper, we will extend this topological truth into transfinite analysis by requiring that infinitesimal information encoded on the smaller sphere can be algorithmically permuted with infinite (divergent) information on the larger sphere, as in figure

4.

In figure 1, we have coordinates $(\mathrm{Re}(\mathcal{Z}), \mathrm{Im}(\mathcal{Z}))$ and on the 2-sphere we have the polar angles (θ, ϕ). The 2-spherical polar angles are defined such that

$$\theta \in [0, \pi] \quad , \qquad \text{and} \qquad \phi \in (-\pi, \pi] \quad , \tag{12}$$

and we mention these specific coordinates because they demonstrate the quirky features that arise when any infinite plane is mapped onto a finite sphere with conformal coordinates. The map from the extended complex plane to the 2-sphere is injective but the inverse map from $\mathbb{S}^2$ back to the plane is not even a well-defined function. When one asks which planar point is the target of the polar point $\theta = \pi$ under the inverse map, one quickly realizes that the question makes no sense. To get the properly bijective map we make the change $\mathbb{S}^2 \to \mathbb{S}^R$ (which is the largest region of $\mathbb{S}^2$ that can be covered with a single chart.) In developing the argument presented here, we will make great use of a bijective map between planar and spherical representations of $\mathbb{C}^Z$ (which is $\mathbb{S}^R$ with one point removed.)

The infinite extent of the Cartesian coordinates in figure 1, stretching from the origin out to infinity, is condensed onto the surface of the 2-sphere with conformal coordinates whose precise definitions are irrelevant. All that is required to move the domain of $\zeta(\mathcal{Z})$ onto the sphere is to make a conformal change of coordinates in the ordinary fashion. When the Riemann sphere is constructed from planar $\mathbb{C}$, the endpoints of the real and imaginary lines are mapped to $\theta = \pi$. To extend the complex plane all the way into hypercomplexity, which is the purpose of this paper, we will use $\hat{\infty}_4$ instead of $\hat{\infty}_2$ so, in some sense, we are adding the point at infinity to each of four orthogonal directions and then condensing their eight endpoints to the polar point of $\mathbb{S}^2$ as did Riemann with the four endpoints of the real and imaginary axes.

It is our belief that the four-to-one multiplicity here described is certainly related to the ontological resolution of the identity [1]

$$\hat{1} = \frac{1}{4\pi}\,\hat{\pi} - \frac{\varphi}{4}\,\hat{\Phi} + \frac{1}{8}\,\hat{2} - \frac{i}{4}\,\hat{i} \quad , \tag{13}$$

where the $\hat{\Phi}$ term is negative through an intuitive definition of the golden ratio

$$\Phi = \frac{1+\sqrt{5}}{2} \quad , \qquad \text{and} \qquad \varphi = \frac{1-\sqrt{5}}{2} \quad . \tag{14}$$

Certainly this connection between the ring at infinity and the ontological resolution of the identity is evocative of the idea that "the ring is unity" but we only mention it in advance of a few qualitative comments that we will make later.

We have defined all of the above objects in preparation for an argument which will use concepts from the hyperreal number system $^*\mathbb{R}$. The real and complex lines exist

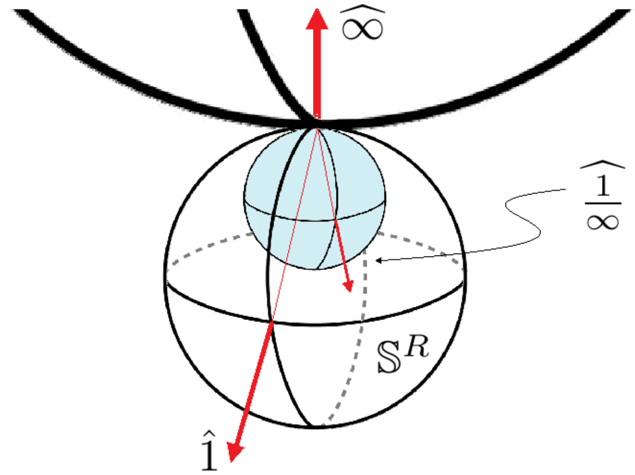

FIG. 4. This figure demonstrates what is meant by an analytic continuation of planar $\mathbb{C}$ into bulk $^\star\mathbb{C}$.

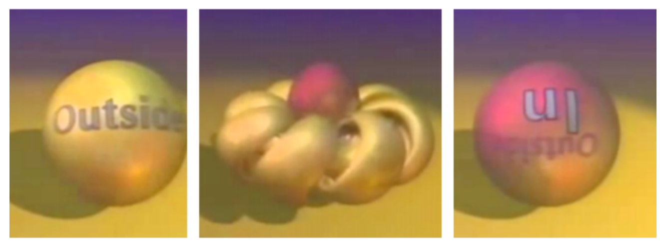

FIG. 5. An illustration of the surprising, non-intuitive result that $\mathbb{S}^2$ is smoothly invertible. This result has been extended to the N-sphere as the sphere theorem which follows from Perelman's (and Hamilton's) proof of the Poincaré conjecture.

on one tier of infinitude and we add two more tiers of infinitude according to figure 4: $\widehat{\infty}$ and $\widehat{1/\infty}$. The reals $\mathbb{R}$ are extended to complex numbers $\mathbb{C}$ with

$$z \to \alpha\hat{1} + \beta\hat{i} \quad , \quad \text{with} \quad \alpha, \beta \in \mathbb{R} \quad , \tag{15}$$

and, using very preliminary formalism, we will extend to hypercomplex numbers $^\star\mathbb{C}$ with

$$z \to \alpha\hat{1} + \beta\hat{i} + \upsilon\hat{\varepsilon} + \chi\hat{\infty} \quad , \quad \text{with} \quad \upsilon \in \mathbb{C}, \chi \in \hat{\mathbb{C}} \quad . \tag{16}$$

To understand equations (15) and (16), consider that, in figure 4, $\hat{1}$ and $\hat{i}$, are on the surface of $\mathbb{S}^R$, $\hat{\varepsilon}$ points to the interior, and $\hat{\infty}$ points to the exterior but we will not use the $\{\hat{\varepsilon}, \hat{\infty}\}$ notation going forward. We will use notation of the form

$$\{\hat{\varepsilon}, \hat{\infty}\} \quad \longrightarrow \quad \{\widehat{1/\infty}, \widehat{\infty}\} \quad \longrightarrow \quad \{\hat{\Phi}^{n-1}, \hat{\Phi}^{n+1}\} \quad . \tag{17}$$

Every real analytical argument will have a corresponding topological framework, and any transfinite, complex analytic continuation of $\mathbb{R}$ must have a sufficiently complex topology that modularizes its finite and transfinite sectors. Without topological modularization, such as the separation of the interior and exterior of a sphere, all elements of order unity will always wash out elements of order ε, elements of order ∞ will always wash out everything, and we would have no way to differentiate $\aleph_0$ from $\aleph_N$, or any of the other kinds of infinities that one might define.

Among infinities, $\aleph_0$ is the smallest possible countable infinity and $\aleph_N$ refers to a set of a N countably infinite infinites $\{(\aleph_0)_1, (\aleph_0)_2, ..., (\aleph_0)_N\}$. Regarding larger infinity, $\hat{\Phi}^n$ can indicate successive tiers of hypercomplex infinitude such that the entire infinite expanse of $\mathbb{R}$ on $\hat{\Phi}^n$ is on the order of a cut in $\mathbb{R}\hat{\Phi}^{n+1}$. $\aleph_N$ and $\hat{\Phi}^n$ are two very different concepts of transfinite infinity and it is the latter whose tiers of infinitude we will call levels of $\aleph$ [2]. The difference in the character of the two infinities is qualitatively demonstrated as

$$\aleph_3 := 3 \times \aleph_0 \quad , \qquad \text{and} \qquad \hat{\Phi}^3 := \aleph_0^3 \quad , \tag{18}$$

and we ignore the implication $\aleph_1 \equiv \aleph_0$ because we will only be using the $\hat{\Phi}^n$ notation. Perhaps $\aleph_0$ is an object in set theory and $\aleph_1$ is an object in multiset theory.

Beginning to make precise hypercomplex definitions, if we say that $\hat{\infty}_2$ is the ring at infinity around the plane that exists on the n^{th} level of $\aleph$ then the $\widehat{\infty}$ and $\widehat{1/\infty}$ shown in figure 4 exist on the $\hat{\Phi}^{n+1}$ and $\hat{\Phi}^{n-1}$ levels of $\aleph$ respectively. Real analysis has no tools to distinguish one type of infinity from another so the only infinity relevant to real analysis is ∞. To make the extension of finite real analysis into some transfinite continuation $\mathbb{R} \to \mathbb{C} \to {}^\star\mathbb{C}$, we need to make some analytical definitions and there must exist a sufficient topological framework for them. Whatever that topology is, it must have an element

$$\hat{\mathbb{R}} \quad \equiv \quad \mathbb{R} \cup \{\infty\} \quad . \tag{19}$$

If the topology required by the new definitions is not reducible to a generic form

$$\mathbb{C}^X \quad \equiv \quad \hat{\mathbb{R}} \cup X \quad , \tag{20}$$

then the definitions will amount to what Laithewaite has called "the multiplication of bananas by umbrellas," meaning that the definitions are contrived.

We suggest that the inability to demonstrate the existence or non-existence of non-trivial zeros of ζ off the critical line may be due to the idea that correct definitions on $\mathbb{C}^Z$ become contrived definitions on $\hat{\mathbb{C}}$ or $\mathbb{S}^R$. All non-contrived definitions will have a valid topology, and, perhaps, it is the condensing of two types of null point into one that causes a paradoxical reliance on some non-spherical topology. We will argue that, by maintaining all null points separately, we can use the spherical $\mathbb{C}^Z \equiv \mathbb{S}^2/\mathbb{S}^0$ topology to add a simplifying symmetry to the framework of analysis. By "maintaining them separately" we mean that they must be defined to have different topologies. As an example, it is possible to consider two null points separately in a way such that the system is described by two independent objects on a shared $\mathbb{S}^R$ topology, and this is what we will be careful to avoid.

We can see that Riemann's definition of analytic continuation meets the topological requirement of equation (20) with

$$\hat{\mathbb{C}} \quad \equiv \quad \hat{\mathbb{R}} \cup \{\hat{\imath}\} \quad , \tag{21}$$

but it does not meet the modularization requirement. When everything lives on the surface of one sphere, there is no way to distinguish one kind of infinity from another. For this reason, we will say that hypercomplex analysis is truly transfinite whereas extended complex analysis is only infinite. The modular topology will be the critical element in pushing from the finite, to the infinite, and then beyond, into the transfinite.

A typical analytic continuation of the elements of real analysis $\{0, 1; \div\}$ into some framework of transfinite analysis begins as

$$\frac{1}{0} \quad \equiv \quad \text{"Undefined"} \quad \mapsto \quad \text{New Defintion} \quad . \tag{22}$$

As we have just made clear, all new definitions are constrained by our ability or inability to define a sufficient topological complex which will preserve the logical predicate of our new definitions without causing a paradox. One such paradox would be that if we put infinity next to a finite quantity in the same modular sector then the infinite will always eat the finite. For example, consider $\vec{r}$ such that

$$\vec{r} = \big(53\infty^2 - \infty\big)\hat{x} + \big(\infty^{762}\big)\,\hat{y} + \left(7\varepsilon + \frac{\infty}{137}\right)\hat{z} \tag{23}$$

$$= \infty\hat{x} + \infty\hat{y} + \infty\hat{z} \quad . \tag{24}$$

The system $\{\hat{x}, \hat{y}, \hat{z}\}$ represents a globally modular topology but within each modular sector $\hat{e}_i$ there is no nested modular structure. We have no way to differentiate ∞^2 from any other permutations of infinity that appear. As a workaround, we will define a new class of transfinite infinities and label them with $\hat{\Phi}^n$ so that $\vec{r}$ can be represented as

$$\vec{r} = \big(53\hat{\Phi}^2 - \hat{\Phi}\big)\hat{x} + \big(\hat{\Phi}^{762}\big)\hat{y} + \left(7\hat{\Phi}^{-1} + \frac{1}{137}\hat{\Phi}\right)\hat{z} \tag{25}$$

$$= \left(\frac{1}{137}\hat{z} - \hat{x}\right)\hat{\Phi} + \big(53\hat{x}\big)\hat{\Phi}^2 + \big(\hat{y}\big)\hat{\Phi}^{762} + ...$$

$$... + \big(7\hat{z}\big)\hat{\Phi}^{-1} \quad . \tag{26}$$

Here, we will make a definition that something is only transfinite if it spans more than two levels of $\aleph$. We say "more than two" because the analytical framework that we have criticized above as being only infinite but not transfinite can be said to span two levels of $\aleph$ if one replaces the ∞ symbol with $\hat{\Phi}^2$ and writes the identity as $-\varphi\hat{\Phi}$. Hypercomplex infinity $\hat{\infty}_4$ has four orthogonal channels but only spans three levels of $\aleph$ so

$$\{\varepsilon,\{1,i\},\infty\} \quad\longrightarrow\quad \{\hat{\Phi}^{n-1},\hat{\Phi}^{n},\hat{\Phi}^{n+1}\} \quad . \tag{27}$$

To extend finite real analysis into the smallest possible infinite complex analysis $\hat{\mathbb{C}}$, we need $\hat{i}$, and we need to take the undefined element of real analysis $1/0$ and make a definition for it like

$$\hat{\infty}_1 \quad\equiv\quad \frac{1}{0} \quad . \tag{28}$$

Riemann extended $\mathbb{R}$ to $\hat{\mathbb{C}}$ with $\hat{\infty}_2$ which implicitly contains ∞ and $i\infty$. However, to get compliance with what we have required in equation (20) we should only consider ∞ added to the end of r in the plane polar coordinates (r,θ) rather than to the end of both the real and imaginary lines in the rectangular coordinates. Then we can take the definition

$$\hat{\infty}_2 \quad\equiv\quad \{\hat{i},\hat{\infty}_1\} \quad , \tag{29}$$

so that equation (6) is replaced with

$$\hat{\mathbb{C}} \quad\equiv\quad \mathbb{R}\cup\{\hat{\infty}_2\} \quad . \tag{30}$$

By going into bulk hypercomplexity with $\hat{\mathbb{C}}\rightarrow{}^\star\mathbb{C}$, as in figure 4, we introduce four channels with

$$\hat{\infty}_4 \quad\equiv\quad \{\hat{i},\hat{\infty}_1,\widehat{\infty},\widehat{1/\infty}\} \quad , \tag{31}$$

so that

$${}^\star\mathbb{C} \quad\equiv\quad \mathbb{R}\cup\{\hat{\infty}_2\}\cup\{\widehat{\infty}\}\cup\{\widehat{1/\infty}\} \quad . \tag{32}$$

${}^\star\mathbb{C}$ clearly conforms to equation (20). Here, $\widehat{\infty}$ refers to "infinity bigger than infinity" which is a concept that can only be analyzed with rigor on a modular topology but should be familiar from the analysis of ${}^*\mathbb{R}$.

Riemann complexified real analysis with complex numbers on the extended complex plane and we hypercomplexify the 2-sphere by extending the neighborhood around every point in $\mathbb{S}^2$ into the bulk, as in figure 4. Riemann used $\hat{\infty}_2$ for his analytic continuation of $\mathbb{R}$ and we will use $\hat{\infty}_4$ instead. This new hypercomplex infinity contains $\hat{\infty}_2$, and adds the hyperreal-valued infinite and infinitesimal elements $\widehat{\infty}$ and $\widehat{1/\infty}$. Hypercomplexity shall differ from infinite complexity through the restriction of infinity to only three simultaneous tiers of infinitude, or levels of $\aleph$, that shall be denoted as

$$\hat{\Phi}^{n+1}=\hat{\Phi}^{n}+\hat{\Phi}^{n-1} \quad , \qquad \text{with} \qquad n\in\mathbb{Z} \quad . \tag{33}$$

With three levels of $\aleph$, we expect

$$\widehat{\infty} \quad := \quad \hat{\Phi}^{n+1} \tag{34}$$

$$\hat{1} \quad := \quad \hat{\Phi}^{n} \tag{35}$$

$$\widehat{\frac{1}{\infty}} \quad := \quad \hat{\Phi}^{n-1} \quad , \tag{36}$$

and we also expect the most important form of equation (33) to be $\hat{\Phi}^2=\hat{\Phi}+\hat{1}$. This is an interesting case because $n=1$ makes the LHS of equation (35) exactly identical with the RHS of equation (36). In the complex analysis that doesn't know about $\hat{\Phi}^n$, we have corresponding definitions like

$$\widehat{\infty} \quad\cong\quad \infty \tag{37}$$

$$\hat{1} \quad\cong\quad \mathcal{Z} \tag{38}$$

$$\widehat{\frac{1}{\infty}} \quad\cong\quad d\mathcal{Z} \quad . \tag{39}$$

These equations demonstrate an important symmetry because equation (38) says that $\mathcal{Z}$ is like $\hat{1}$ and not like $\widehat{\infty}$ or $\widehat{1/\infty}$, but equation (35) says $\hat{1}$ is like any random $\hat{\Phi}^n$. This is the beginning of a paradox, and paradox is often the beginning of proof.

Searches for zeros off the critical line can only approach null points in the domain of ζ by finite increments. Therefore, when we consider any point p close to a null point, there are infinitely many other points that are closer to the null point than p. It is the utility of hypercomplex analysis that we can define other points P that are immediately adjacent to a given null point in the finite analysis of a certain level of $\aleph$. Near the polar singularity $\theta=\pi$, infinitely complex points are defined as

$$(\theta,\phi)=\left(\pi+\sum_{j=1}^{N}\varepsilon^j\digamma_j \ , \ \phi_0+\sum_{k=1}^{M}\varepsilon^k\digamma_k\right) \quad . \tag{40}$$

For ease of analysis, we are defining hypercomplexity to be the limit of infinite complexity where $N=M=2$, as in equation (33). For further ease, we will consider two lower levels of $\aleph$ rather than a lower one and a higher one and this looks like

$$\hat{1} \quad \cong \quad \mathcal{Z} \tag{41}$$

$$\widehat{\frac{1}{\infty}} \quad \cong \quad d\mathcal{Z} \tag{42}$$

$$\frac{\widehat{1/\infty}}{\infty} \quad \cong \quad \big(d\mathcal{Z}\big)^2 \quad . \tag{43}$$

In this way, we hope to simplify things by removing the object ∞ from the RHS definitions. Terms of order $(d\mathcal{Z})^2$ are completely mundane and ∞ is essentially impossible to work with directly. What we have done here is to translate our analysis by one discrete unit of $\aleph$ which is $\hat{\Phi}$. To understand how this might be useful, consider that when using the sphere theorem to permute the interior and exterior of a sphere, such can happen fully within a larger sphere whose interior remains its interior throughout the rearrangement of the interior and exterior of the smaller sphere. (We only mention the sphere theorem in this paper because it implies the existence of a calculus based argument dual to our mostly geometric argument.)

Consider one sphere that exists by itself. When the interior is swapped with the exterior, the finite interior region is rarefied to spread over all of exterior space. Now consider that sphere situated on the interior of another sphere. When the interior and exterior of the inner sphere swap, two finite elements get mapped to two finite elements. For applications that involve only the surface of the sphere and a pair of arrows pointing "in" and "out," the problem of one sphere is exactly the same as the problem of one sphere embedded in another but, with the embedding, all of our maps are finite-to-finite. In the former situation, we had one finite-to-infinite map and another infinite-to-finite one, and both of these types of maps present a lot of problems for analysis. Recall that the finite-to-infinite map $\zeta : 1 \mapsto \infty$ has driven a large part of the present discussions.

Equations (34-36) are formatted for consistency with equations (41-43) as

$$\hat{1} \quad := \quad \hat{\Phi}^0 \tag{44}$$

$$\widehat{\frac{1}{\infty}} \quad := \quad \hat{\Phi}^1 \tag{45}$$

$$\frac{\widehat{1/\infty}}{\infty} \quad := \quad \hat{\Phi}^2 \quad , \tag{46}$$

where we have reversed the (arbitrary) direction of increasing n with respect to increasing infinitude. Now higher n means a lower level of $\aleph$.

Moving away from the specific point $\theta = \pi$ specified by equation (40) we may define the hypercomplex coordinates around an arbitrary point as

$$(\Delta\theta, \Delta\phi) = \left(\sum_{j=1}^{2} \hat{\Phi}^j F_j \ , \ \sum_{k=1}^{2} \hat{\Phi}^k F_k \right) \tag{47}$$

$$= \Big(F_1^\theta \hat{\Phi} + F_2^\theta \hat{\Phi}^2 \ , \ F_1^\phi \hat{\Phi} + F_2^\phi \hat{\Phi}^2 \Big) \quad . \tag{48}$$

Here, we have to ask an important question. Are the $\hat{\Phi}^n$ that appear in $\Delta\theta$ the same ones that appear in $\Delta\phi$? The answer is yes. F_1^θ and F_1^ϕ specify a point on the surface of the first infinitesimal sphere, which is the small sphere in figure 4, and (F_2^θ, F_2^ϕ) is a point on a second nested infinitesimal sphere written in the doubly infinitesimal coordinates $(\theta\hat{\Phi}^2, \phi\hat{\Phi}^2)$ that live there (not shown in figure 4.) This is all well and good but it begs a second question. Even if they are the same, do they have to be? In that case, the answer is no and it gives us some guidance about how we can define two different types of singularity on the same sphere.

Among analytic and coordinate singularities we can say that one goes "in" and the other goes "out," or we can even take combinations of $\{\widehat{\infty}, \widehat{1/\infty}\}$ as either "mostly in" or "mostly out." Furthermore, we can say that the answer to our first question was yes because the $\hat{\Phi}^n$ were attached to the same null point (θ, ϕ), but that the $\hat{\Phi}^n$ attached to some other point (θ', ϕ') exist completely independently. This will mean that every point has its own uniquely associated transfinite structure, but there is something particular about $\hat{\Phi}^0 \equiv \hat{1}$ which all points of $\mathbb{C}^Z$ must have in common if they are to constitute a coherent sphere. If every point was disconnected on every level of $\aleph$ then we would have no continuum.

Here we will have a brief aside about physics and the theory of infinite complexity, and how it might relate to the Riemann zeta function. When the $\{\hat{\Phi}, \hat{\Phi}^2\}$ in $\Delta\theta$ are different than those in $\Delta\phi$, for any reason, not only the one proposed above, then the hypercomplex coordinates become

$$\big(\Delta\theta, \Delta\phi\big) \quad \longrightarrow \quad \big(\Delta\theta_+, \ \Delta\theta_-, \ \Delta\phi_+, \ \Delta\phi_-\big) \quad , \tag{49}$$

and it looks very much like something that would be written in the ontological basis $\{\hat{i}, \hat{\Phi}, \hat{2}, \hat{\pi}\}$. Let's consider that basis in terms of an interpretation of numbers suggested by Gauss. Positive numbers shall be called direct, negative numbers inverse, and imaginary numbers shall be called lateral. To that end, consider two groupings of equation (13)

$$\hat{1} = \left(\frac{1}{4\pi}\, \hat{\pi} - \frac{\varphi}{4}\, \hat{\Phi} \right) - \left(\frac{i}{4}\, \hat{\imath} - \frac{1}{8}\, \hat{2} \right) \quad , \tag{50}$$

and

$$\hat{1} = \left(\frac{1}{4\pi}\,\hat{\pi} - \frac{i}{4}\,\hat{i} \right) - \left(\frac{\varphi}{4}\,\hat{\Phi} - \frac{1}{8}\,\hat{2} \right) \quad . \tag{51}$$

In Gaussian terminology, each grouping of the resolution of the identity has a direct parenthetical term and an inverse parenthetical term. Within each of those two terms, there exists another pair of direct and inverse (positive and negative) terms, and we can also compare this to equation (2) in ordinary terms of what it means to be direct or inverse. When direct means "numerator" and inverse means "denominator" the analytical structure of ζ exhibits the exact same nested structure as the ontological resolution of the identity. Considering equation (2), perhaps the direct parenthetical term from equation (50) or (51) is on the top of ζ and the inverse term is on bottom, and the denominator in each integrand is controlled by the inverse term that appears inside each parenthetical term of the grouped resolution of the identity.

We have developed the ontological basis in support of other efforts to unify gravitation with quantum mechanics [1–7] but somehow we have been able to do almost all of it with just $\hat{\pi}$ and $\hat{\Phi}$. In that work, we have shown little reliance on $\hat{i}$ and $\hat{2}$ so if they are equally important members of the ontological basis $\{\hat{i},\hat{\Phi},\hat{2},\hat{\pi}\}$ then why aren't they used as prominently? The answer is that we have only treated non-relativistic quantum theory which is the limit of QFT where the third and fourth components of the Dirac vector vanish because they depend on the relativity parameter $\beta=(v/c)^2$. If we associate $\hat{2}$ and $\hat{i}$ with these vanishing parts of the Dirac bispinor then it makes perfect sense that we would be able to fully describe the nonrelativistic limit of the theory without them. However, in terms of pure analysis, even when an analytical $\beta=0$ constraint is imposed, the topology that allows four channels doesn't go away. In physics it is normal to assume a minimal topology, but for analysis we should consider that all four channels are always there, even when two of them tend to zero.

The Gaussian interpretation of our resolution of the identity demonstrates the exact spinor-bispinor form of the Dirac vector, and even ζ itself shows this structure in the Gaussian interpretation. The ontological basis vectors that we have chosen as not contributing to nonrelativistic quantum theory, $\hat{2}$ and $\hat{i}$, appear as the inverse terms to the direct terms $\hat{\Phi}$ and $\hat{\pi}$ in equation (51) and are the entire inverse parenthetical term in equation (50). Furthermore, every non-trivial zero of $\zeta(\mathcal{Z})$ is symmetric about the real line with another zero and if there are any zeros off the critical line then they will have a symmetric zero on the other side of the critical line. This means that non-trivial zeros always appear in pairs, and if there is one off the critical line then it will be symmetric with three other zeros. Therefore, if they exist, $\bar{\mathcal{Z}}_{\bar{n}}$ will always appear in pairs of pairs. Now that we have taken note of these features, back to the Riemann hypothesis.

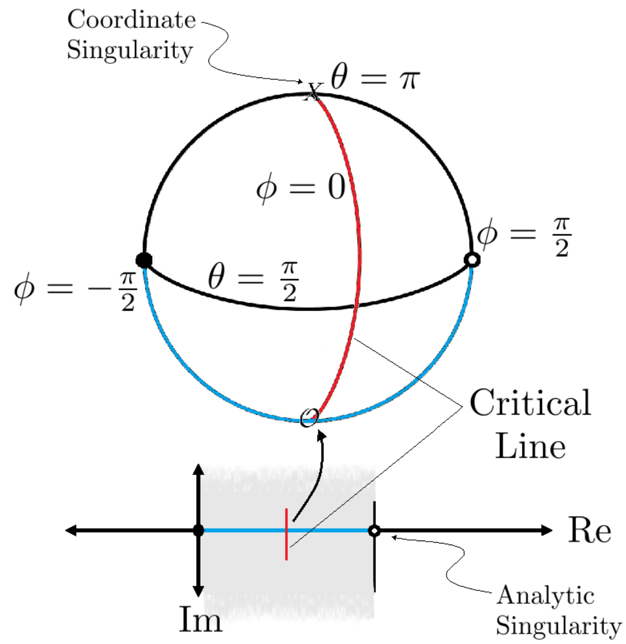

FIG. 6. The planar domain of ζ is mapped onto a sphere. The preimage of the origin of coordinates $\mathcal{O}$ on the sphere is the planar point $\mathcal{Z}=1/2$.

Consider the map from the plane to the sphere shown in figure 6. The origin of the planar coordinates is shown with a black dot and the singularity at $\mathcal{Z}=1$ is shown with a white dot. The origin of coordinates on the sphere is marked with $\mathcal{O}$ and the sphere's coordinate singularity is marked with X. The planar points of interest are mapped to polar points as

$$\mathcal{Z}=0 \quad\mapsto\quad \big(\theta,\phi\big)=\big(\tfrac{\pi}{2},-\tfrac{\pi}{2}\big) \tag{52}$$

$$\mathcal{Z}=1 \quad\mapsto\quad \big(\theta,\phi\big)=\big(\tfrac{\pi}{2},\tfrac{\pi}{2}\big) \tag{53}$$

$$\mathcal{Z}=\frac{1}{2} \quad\mapsto\quad \big(\theta,\phi\big)=\big(0,\text{multivalued}\big) \tag{54}$$

$$\mathcal{Z}=\pm\infty \quad\mapsto\quad \big(\theta,\phi\big)=\big(\pi,\text{multivalued}\big) \tag{55}$$

$$\mathcal{Z}=\pm i\infty \quad\mapsto\quad \big(\theta,\phi\big)=\big(\pi,\text{multivalued}\big) \quad , \tag{56}$$

and, here, the reader should be careful not to assume that the multivaluedness of the coordinates is only a property of the sphere. When we write the complex number $\mathcal{Z}$ in terms of the plane polar coordinates (r,θ), we see the same behavior with

$$\mathcal{Z}=0 \quad\mapsto\quad \big(r,\theta\big)=\big(0,\text{multivalued}\big) \quad . \tag{57}$$

The key features to note in figure 6 are that the critical line becomes the $\phi = \{0, \pi\}$ meridian, the hemisphere toward the reader is the upper complex half-plane, and the hemisphere away from the reader (not shown) is the lower complex half-plane. We send the planar real line to the two great semicircles $\phi = \pm\pi/2$. Whatever the conformal coordinates are, we have chosen them so that the real line between zero and one is mapped onto the lower semicircle in the plane of the page.

Why have we chosen the specific map in figure 6? We might have put the origin of the plane at the origin of the sphere, or we might have put the analytic singularity there but, instead, we have chosen to send planar $\mathcal{Z} = 1/2$ to $\theta = 0$. To see our motivations, consider equation (2) and note that ζ is technically a function of $\mathcal{Z} - 1$. $\mathcal{Z}$ is constructed directly from the z that appears in equation (1) so that the origins of $\mathbb{R}$ and $\mathbb{C}$ are collocated. However, since ζ only depends on $\mathcal{Z} - 1$, that means $\zeta(\mathcal{Z})$ is not a properly irreducible representation of the continuation of $\mathcal{D}(z)$. Written in irreducible form we have

$$\zeta(\mathrm{Z}) = \left(\frac{\displaystyle\int_0^\infty \frac{x^{\mathrm{Z}}}{e^x - 1}\, dx}{\displaystyle\int_0^\infty \frac{x^{\mathrm{Z}}}{e^x}\, dx} \right) \quad , \quad \text{with} \quad \mathrm{Z} = \mathcal{Z} - 1 \ . \tag{58}$$

Among the several choices for points to map to $\mathcal{O}$, we have chosen the critical line as the most important feature and if one wishes to argue against the argument presented here then the specific choice of the map in figure 6 might be a good starting point. Nevertheless, this is the map we will use in the present argument.

Figure 7 shows roughly how the critical strip appears on the sphere in the chosen conformal coordinates and figure 8 shows the polar region around $\theta = \pi$. In figure 8, we see that the lines all approach the null point at $\theta = \pi$. Since these are all straight lines in the plane, they approach planar infinity and, therefore, they must approach the polar null point after mapping to the sphere. In figure 8, the purpose of the X, in addition to marking $\theta = \pi$, is to remind the reader that the lines do not actually terminate on the point, they merely approach it. In figure 9, the lines do terminate on the circle but their endpoints are separated by non-vanishing hypercomplex infinitesimal elements $\{\eta_1, \eta_2\}$. Also note that the point $\theta = \pi$ is multivalued in ϕ but if we consider an infinitesimal disk instead then each point on its perimeter can be identified with a value from $\phi \in (-\pi, \pi]$. The reader should further note that all the lines in figure 9 point directly to the center of the circle even though the figure does not show that. Instead, the figure demonstrates that the three lines $\mathrm{Re}(\mathcal{Z}) = \{0,\ 1/2,\ 1\}$ become approximately parallel near the pole. In extended complex analysis, we can make true statements regarding figure 9 like

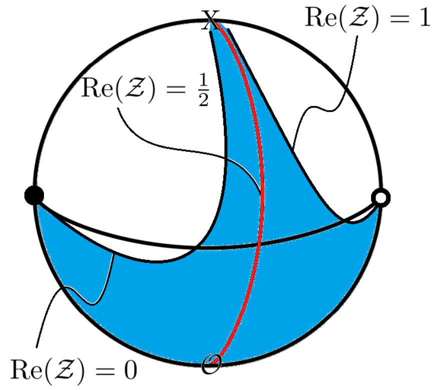

FIG. 7. This figure shows how the critical strip is arranged on the surface of the sphere. There are many other possible arrangements but this is the one presently considered.

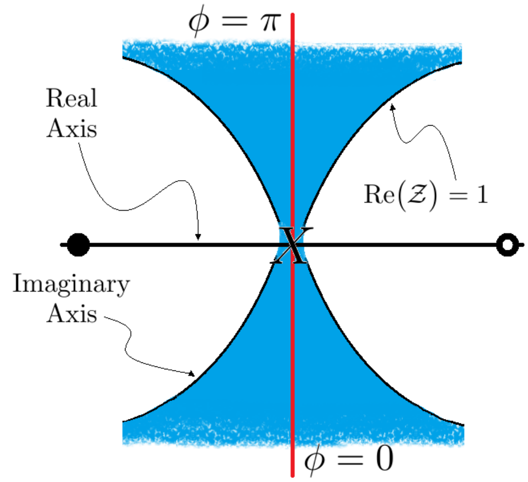

FIG. 8. This figure shows the region around what has been the top of the sphere in previous figures.

$$\eta_1 = 0 \quad , \qquad \text{and} \qquad \eta_2 = 0 \quad , \tag{59}$$

but, in hypercomplex analysis, there is a clear, important distinction between

$$\eta_1 \circ \hat{\Phi}^0 = \varepsilon \quad , \qquad \text{and} \qquad \eta_2 \circ \hat{\Phi}^0 = \varepsilon \quad , \tag{60}$$

and

$$\eta_1 \circ \hat{\Phi}^1 = \text{finite} \quad , \qquad \text{and} \qquad \eta_2 \circ \hat{\Phi}^1 = \text{finite} \quad . \tag{61}$$

Here, we will show why $\mathbb{C}^Z$ is absolutely critical for our argument against Riemann's hypothesis, and that we could not possibly hope to make it using $\mathbb{S}^R$. Consider two Riemann spheres $\mathbb{S}^R$, one of order unity on $\hat{\Phi}^0$ and an infinitesimal one embedded inside at $\hat{\Phi}^1$ so that the null points of each sphere are at the same north polar point, as in figure 4. Certainly we can create figure 9 via smooth deformation of the hypercomplexly infinitesimal neighborhood around that shared null point. As we widen that neighborhood into a finite region on the larger sphere, the smaller sphere will be pulled up into that space and flattened. Certainly it easy to see how the material around the perimeter of figure 9 is part of the larger sphere, and the inside of the circle is the complete surface of the smaller sphere stretched out for visualization purposes. With $\mathbb{S}^R$, if we wanted to consider a third level of $\aleph$, as in figure 10, we would have to punch a hole in the $\hat{\Phi}^1$ sphere where none exist.

When we map the plane to the sphere, the ring at infinity becomes a point at the pole and what we have described above is the reverse process. By spreading out the null point onto a finite region, the point becomes a ring. When the single null point on the first infinitesimal Riemann sphere $\mathbb{S}^R\hat{\Phi}^1$ is the ring shown in figure 9, there is no other null point for us to spread over a smaller but still finite region at the center of the $\hat{\Phi}^1$ circle. However, if we replace $\mathbb{S}^R$ with $\mathbb{C}^Z$ then one of the infinitesimal sphere's null points is the ring in figure 9 and there will be a second one in the center. This would be the south pole of the smaller sphere that gets pulled directly upward when the north pole is rarefied directly outward in all directions according to the operation that creates the representation in figure 9. When we have that second null point, we can spread its neighborhood out into another finite region as in figure 10, but if we had been working with $\mathbb{S}^R$ then we would have had to punch a hole in the topology which is not allowed in the conformal transformations that we are considering.

Now we have an interesting interpretation of the mathematics that says all of the points in the $\hat{\Phi}^1$ region lie within the domain of $\zeta(\mathcal{Z})$. The coordinates of the points in the $\hat{\Phi}^1$ region are all of the form

$$\big(\theta,\phi\big) = \big(\pi+\Delta\theta, \phi_0+\Delta\phi\big) \quad , \quad \text{with} \quad \pi+\Delta\theta<\pi \quad , \tag{62}$$

so this means none of the points in the $\hat{\Phi}^1$ region are the polar singularity $\theta=\pi$ and we can say this about the points in the $\hat{\Phi}^n$ region. Therefore, in precise language, we should say that the interior of the $\hat{\Phi}^1$ circle is a hypercomplexly infinitesimal neighborhood of $\theta=\pi$ rather than the point itself but that, if we were to consider only points on the order of $\hat{\Phi}^0$, the circle and the point are the same. The important distinction is that we are conducting a hypercomplex analysis not restricted to $\hat{\Phi}^0$ alone. The fact that $\theta=\pi$ is a singularity shall imply that it is irreducibly infinitesimal. If the $\hat{\Phi}^1$ points are in $\mathbb{C}^Z$ then the bijection with the plane must send those points somewhere and, since they are inside the sphere's $\hat{\Phi}^1$ ring, they should be mapped to a planar point beyond infinity. However, our plane is just a plane; it is not yet an extended plane that includes infinity. (This follows because the plane we refer to is $\mathbb{C}^Z$, the domain

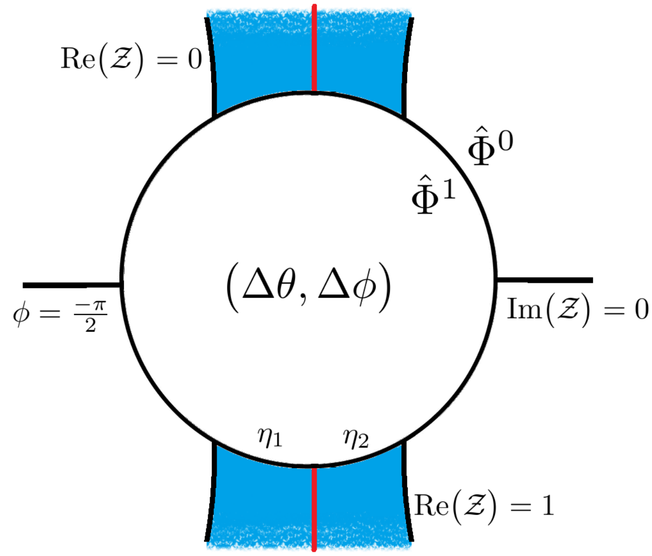

FIG. 9. The $\mathbb{C}^Z$ topology has a hole in it at $\theta=\pi$ and we can spread the hypercomplex neighborhood around it into a finite region to consider what could be happening within.

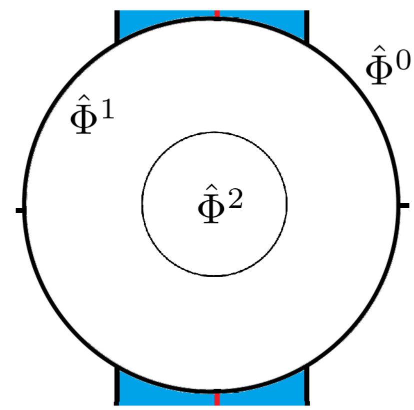

FIG. 10. This figure demonstrates the beginning of an infinite series of nested, expanded, increasingly hypercomplexly infinitesimal neighborhoods around the polar coordinate singularity at $\theta=\pi$.

of ζ, and ζ is not analytic at ∞.) Therefore, we will say that we are using the hypercomplex plane $\mathbb{C}^Z$ which does not include the boundary at uncountable infinity but does include the region between countable and uncountable infinity so that $\zeta(\infty)\to\zeta(\hat{\Phi}^\infty)$ is the point of analytic breakdown but $\zeta(\infty)\to\zeta(\hat{\Phi}^n)$ is allowed. It is clear that, if so desired, we could construct a convergent Taylor series expansion around the point $\mathcal{Z}=\hat{\Phi}^n$.

The critical reader might ask, "A precise number bigger than infinity doesn't make sense as a concept, seems contrived, and even if those points exist, what numbers could we use to describe them?" In that case, we remind the reader of the number $\sqrt{-1}$ that Riemann used in his own analysis. A number that is the square root of a negative number doesn't make any sense, certainly seems contrived, and is, in fact, *completely imaginary!* If Riemann (and everyone else) can simply call that number i then we can likewise call our tiers of transfinite infinitude $\hat{\Phi}^n$. We can answer the critical reader's question about the bijective target point in the plane using the notation from equation (48). Under bijection, the planar identity of the point is

$$\big(\Delta\theta,\Delta\phi\big)=\big(F_1^\theta\hat{\Phi}^1,F_1^\phi\hat{\Phi}^1\big)\quad\mapsto\quad\mathcal{Z}=\big(z+i\gamma\big)\hat{\Phi}^1\ . \tag{63}$$

This completely solves the problem because we can keep adding $\hat{\Phi}^n$ without ever getting to the ring at uncountable infinity or, in the spherical representation, the irreducible singularity at $\theta\!=\!\pi$. Interestingly, as we keep adding regions of infinity beyond infinity, the planar representation becomes exactly like the area around the singularity, as in figure 10. Furthermore, if the plane is the surface of a sphere whose radius is infinity, then perhaps the boundary at uncountable infinity is where the curvature becomes non-negligible after being completely negligible through all the $\hat{\Phi}^n$ planar annuli and that, ultimately, we have developed the hypercomplex version of the ordinary inversion map between the two coordinate charts that cover $\mathbb{S}^2$. (Note that if we have two singularities in $\mathbb{C}^Z$, and we put it on $\mathbb{S}^2$, then there will be four singularities associated with it: a pair of singularities in each of a pair of charts.)

If we want to consider the planar representation as different than the spherical representation, then we can consider a stack of planes, each one labeled with $\hat{\Phi}^n$, and let the bijection be between the surface of one hypercomplex sphere and an infinite number of planes. This likely has immediate application to the **Banach–Tarski paradox** because a complete sphere can be constructed from each of the infinitely many planes. Also note that a stack of N discrete planes is an excellent example of what we have called modular topology, and a series of concentric planar annuli also have the requisite modular properties. Finally, note that nothing in our definitions prevents the stacked planes of ascending levels of $\aleph$ from having their own countably transfinite annuli defined according to the objects in equation (18).

Here, we will make a simplistic statement that is wrong (sans serif) to demonstrate a qualitative principle, and then shortly we will show that a slightly different true statement affirms the truth of that principle. Considering figures 9 and 10, the important thing to note is that, regardless of the non-trivial parameterizations of rectangular grid lines in the coordinates on the $\hat{\Phi}^0$ surface of the sphere, once those lines get to the $\hat{\Phi}^1$ circle, they have achieved their asymptotic limits and all point in the radial direction. Therefore, the parameterizations of the lines in the $\hat{\Phi}^1$ region can depend only on θ but not ϕ.

There exists a famous argument about infinity that says a larger circle must have more points than a smaller circle. The argument states if that one traces a ray from the center of a small circle to each point on its circumference, and then continues those rays onto the circumference of a larger concentric circle, then there will be gaps between the rays meaning that the larger circle has more points. There are a number of counter arguments but we will simply require that the points on the $\hat{\Phi}^2$ circle are "infinitesimal points" with respect to those on the $\hat{\Phi}^1$ circle. The circles are self-similar in every way other than that they exist on different levels of $\aleph$. Therefore, we can say that all the nested circles considered have the same number of points; they are merely different kinds of points. Furthermore, it is obvious that there exist smooth analytic continuations of the critical line, the real axis, and the imaginary axis across any number of circles as they tend toward $\theta\!=\!\pi$ forever. Therefore, without even extending the lines through the $\hat{\Phi}^1$ region, we can draw the real and imaginary axes in the $\hat{\Phi}^2$ region. Since the number of points in each circle is the same, and the lines that connect the points of each must lie along some line of constant ϕ, all the points of the $\hat{\Phi}^2$ circle that are in its first quadrant should be connected radially to the points in the first quadrant of the $\hat{\Phi}^1$ circle, and likewise for the other quadrants. Figure 11 shows that we are unable to accommodate the requirement for one-to-one point mapping with purely radial lines.

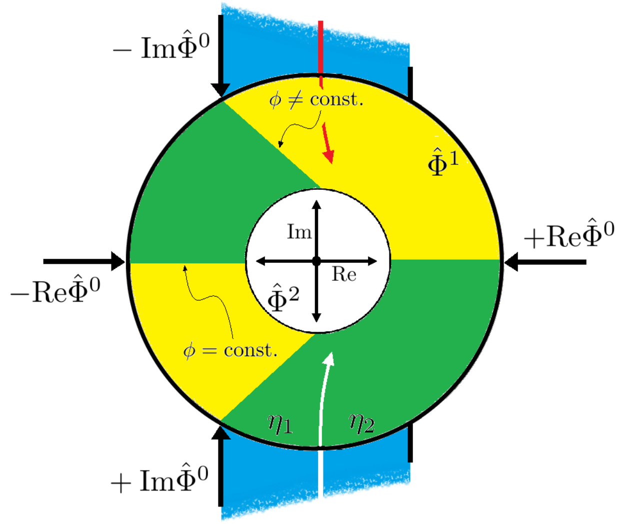

FIG. 11. This figure shows a subtle asymmetry related to the off-center position of the critical strip in $\mathbb{C}$. The critical line is colored white in the dark region for contrast.

The critical reader will ask why we can't just deform the point density in the circumference of the $\hat{\Phi}^2$ circle so that the lines would be radial and we have an excellent answer for that. Due to the argument in figure 3, there can be no preferred azimuthal directionality beyond infinity. Therefore, anything that we create beyond infinity must have perfect rotational symmetry. If we said that the $\hat{\Phi}^2$ circle had more points on one side than the other, that would be the opposite of rotational invariance.

Now we have derived a contradiction that shows the lines must reach their asymptotic radial behavior at the $\hat{\Phi}^2$ circle, not the $\hat{\Phi}^1$ circle as stated above. Since the deviation induced by the choice of map in figure 6 does not have rotational symmetry, the argument about the distribution of points on the $\hat{\Phi}^2$ circle has equal relevance to the directionality of the lines within the $\hat{\Phi}^2$ region. Therefore, the lines must reach their radial asymptotic behavior at $\hat{\Phi}^2$ even if they cannot have reached them at $\hat{\Phi}^1$. Furthermore, it is commonly understood in mathematics that there can be no demonstrable structure between countable and uncountably infinity but, here, we have demonstrated a non-trivial structure in the region between $\hat{\Phi}^1$ and $\hat{\Phi}^\infty$. Above we claimed wrongly the lines will reach their asymptotes at $\hat{\Phi}^1$ but we see that they do not reach them until $\hat{\Phi}^2$.

The reader might ask, "Why isn't the $\hat{\Phi}^1$ region also constrained to have a uniform distribution of points on its circumference? Surely the argument from figure 3 applies to $\hat{\Phi}^1$ and $\hat{\Phi}^2$ equally." It does not. To see this, consider the planar representation of $\mathbb{C}^Z\hat{\Phi}^0$ (figure 1) without the $\hat{\Phi}^n$ annuli out beyond countable planar infinity. Since the critical strip is asymmetrical in this planar region, and we have put the critical line at the origin of spherical coordinates, the conformal coordinates are asymmetrical on the corresponding spherical region, and that region touches the $\hat{\Phi}^1$ circle so its point density is asymmetrical.

Now consider how the points beyond the planar ring at infinity are mapped into the spherical $\hat{\Phi}^1$ region. If we expand the sphere's polar point, or rather the hypercomplex neighborhood around it, into the finite region $\hat{\Phi}^1$, as in figure 9, then we have to get some better representation of the area beyond the ring at planar infinity before we can begin to consider its bijection. We can map an infinite plane to $\mathbb{S}^2$, because we can choose an origin of coordinates, and then let the conformal coordinates take care of the behavior at infinity but, when we are considering a transfinite plane such as the $\hat{\Phi}^1$ planar annulus, we cannot even begin to consider its origin because the entire region is hidden beyond infinity. We have increased the area of the polar point from zero to some finite value where we can see the infinitesimal coordinates so we must decrease the transfinite area of the $\hat{\Phi}^1$ region to be the only-infinite planar area whose bijection with $\mathbb{S}^R$ is well known. We know how to map the plane to a sphere, but we do not know how to map the annulus beyond the ring at planar infinity to a sphere so we need to make a change that puts it into a workable form. This requires shrinking the planar region on the interior of the first annulus to a point. When this is completed, the entire $\hat{\Phi}^0$ real line is compactified to a point that becomes the origin of the infinite, but not transfinite, $\hat{\Phi}^1$ planar region, as in figure 12. Afterwards, with respect to the finite analysis of the bijection, the critical line is perfectly symmetric in the $\hat{\Phi}^1$ plane. This explains why the $\hat{\Phi}^2$ circle is constrained in a way that the $\hat{\Phi}^1$ circle is not. After mapping the $\hat{\Phi}^0$ planar region to the sphere, the global asymmetry of the that region touches the $\hat{\Phi}^1$ circle. However, there are no asymmetric features of the $\hat{\Phi}^1$ planar annulus or, for that matter, any other planar annulus $\hat{\Phi}^n$ with $n>1$. Therefore, both regions touching the $\hat{\Phi}^2$ circle will be populated with perfectly symmetrical maps but this was not true for the $\hat{\Phi}^1$ circle. Also note that it would have been impossible to make this argument without introducing at least a third level of $\aleph$ to make our analysis fully transfinite.

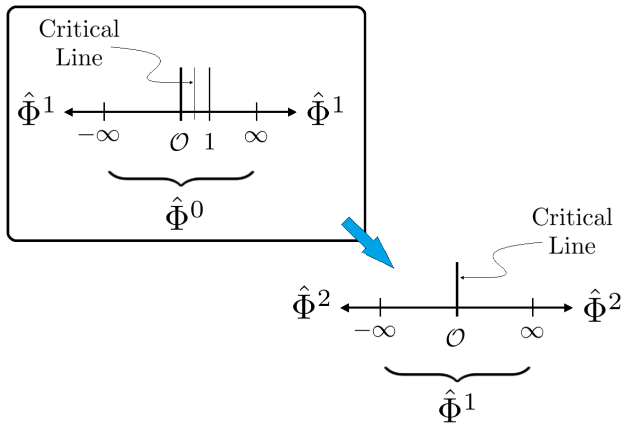

FIG. 12. The critical strip is off-center in the $\hat{\Phi}^0$ region, but is not in the finite analysis of $\hat{\Phi}^1$ region after conversion from a transfinite annulus to an infinite plane.

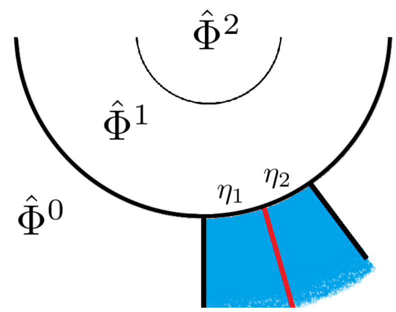

FIG. 13. To accommodate the radial line requirement in the $\hat{\Phi}^1$ region, we may attempt to shift critical strip by η_1.

If we are to force the lines to be strictly radial between the $\hat{\Phi}^1$ and $\hat{\Phi}^2$ circles then we could redraw figure 11, as in figure 13. Since η_1 and η_2 are infinitesimal with respect to $\hat{\Phi}^0$, we might shift the critical line over slightly so as to avoid any paradoxical ϕ dependence in the $\hat{\Phi}^1$ region. However! This creates another contradiction. When the null point is only a point, the azimuthal angle ϕ is multivalued there, as in equations (55) and (56). When the hypercomplex neighborhood around $\theta=\pi$ is deformed into a small circle, then each point on that circle does have a unique ϕ value and, clearly, $\phi\hat{\Phi}^0 = \phi\hat{\Phi}^1$. Then, if figure 11 shows the case where the critical line is the great circle $\phi\hat{\Phi}^0 = \{0,\pi\}$, the critical line in figure 13 can no longer be specified in that way because now the line $\mathrm{Re}(\mathcal{Z})=0$ terminates at $\phi=0$ and they are separated by $\eta_1 \circ \hat{\Phi}^1 = \text{finite}$. The angle is finite in both regions but the associated arc length is only finite in $\hat{\Phi}^1$; it is infinitesimal in $\hat{\Phi}^0$ because of the relatively infinitesimal radius of the $\hat{\Phi}^1$ circle. This is clearly shown in figure 13 but it contradicts the original map from figure 6 which put the critical line on the great circle with $\phi = \{0, \pi\}$. Obviously, any great circle of $\mathbb{S}^2$ passing through the poles of its coordinate system will maintain the same ϕ value across any number of levels of $\aleph$. However, since $\eta_1 \circ \hat{\Phi}^1$ is finite, we see that after shifting the strip over by a infinitesimal distance $\eta_1 \circ \hat{\Phi}^0$, we have changed the ϕ value at which the critical line touches the $\hat{\Phi}^1$ circle by a finite increment.

One might try to argue that, since we have shifted the critical strip by an infinitesimal η_1 with respect to the $\hat{\Phi}^0$ scale, it means that the change in ϕ is also infinitesimal. On closer inspection, one sees that if η_1 is finite with respect to $\hat{\Phi}^1$ then the angular displacement is evidently finite in $\hat{\Phi}^1$ as well. Therefore $\phi\hat{\Phi}^0 = \phi\hat{\Phi}^1$ and the $\phi\hat{\Phi}^0$ values at which the critical line strikes the $\hat{\Phi}^1$ circle must deviate by exactly $\phi\hat{\Phi}^1$ radians from the values $\{0, \pi\}$ it had in figure 11. Then one notices that $\phi\hat{\Phi}^n = \phi\hat{\Phi}^m$ has the implication that there is no such thing as infinitesimal azimuthal angle, but that is also subtly not right. Certainly, we can add infinitesimal azimuth to meridians near the equator to distinguish several hypercomplex meridians sharing a common $\phi\hat{\Phi}^0$ coordinate but we cannot do this near the pole. If we considered infinitesimal $\phi\hat{\Phi}^0$ at the $\hat{\Phi}^1$ circle, it would tell us how the lines that come in from the $\hat{\Phi}^0$ region all terminate on the same point of the $\hat{\Phi}^1$ circle but are actually separated by infinitesimal arc lengths. This is wrong because we have already shown that, with respect to the $\phi\hat{\Phi}^1 = \phi\hat{\Phi}^0$ azimuth, the arc lengths separating the elements of the critical strip are proportional to finite η in $\hat{\Phi}^1$, not infinitesimal ε, as in equation (61). If $\phi\hat{\Phi}^0$ was infinitesimal, then so too would $\phi\hat{\Phi}^1$ be infinitesimal, but then the arc lengths corresponding to the infinitesimal angular displacements would also be infinitesimal, and that contradicts the structure described in equations (60) and (61).

So... we cannot shift the critical strip, as in figure 13, because it changes the ϕ at which the critical line strikes the $\hat{\Phi}^1$ circle and this is not allowed for polar great circles. Polar great circle means "constant ϕ" and the critical line has been defined as such, as in figure 6. If a great circle leaves the south pole with ϕ' but gets to the north pole with $\phi'' \neq \phi'$ then it is not a great circle. We cannot achieve radial lines in the $\hat{\Phi}^1$ region by deforming the density of points on the $\hat{\Phi}^2$ circle because that violates rotational invariance, but there is one more thing we can consider. This has to do with the map chosen in figure 6. If we had mapped $\mathrm{Z} = -1$ to the origin $\mathcal{O}$ of the spherical coordinates instead of $\mathrm{Z} = -1/2$ then the lines would have already been arranged as in figure 13 without us having to do anything.

If the plane is infinite then we can set a sphere on top of any point in the plane and use the polar ray to define a bijective map, as in figure 2. (We can use the linear ray to put the deformed coordinates on the sphere by first doing a conformal transformation in the plane.) If we set the sphere on the point $\{\mathrm{Z}, \mathcal{Z}\} = \{0, 1\}$, that strengthens our argument because it increases the off-centeredness that has to be corrected with non-radial lines in the $\hat{\Phi}^1$ region, as in figure 11. The point $\{\mathrm{Z}, \mathcal{Z}\} = \{-1, 0\}$, which would naturally produce the contradiction-free figure 13, is the only point on the entire plane that allows us to avoid the contradictions we have developed above.

What reason do we have to set the sphere there and to do the ray tracing with respect to that point? Why should we put the sphere at the origin of the $\mathcal{Z}$ (and z) coordinates when the Z coordinates are the ones most natural to ζ? Can we say that if there exists even one map which implies the existence of $\bar{\mathcal{Z}}_{\bar{n}}$ then they must exist, as in equation (5)? That question certainly begs another: if there is only one map that does not imply the existence of $\bar{\mathcal{Z}}_{\bar{n}}$ then why should we prefer that one over infinitely many others that do imply $\bar{\mathcal{Z}}_{\bar{n}}$? (This implication is shown below.) If it is true that we should use the one singular map that negates our argument, the reason would be that we are considering an analytic continuation of the real line z, from equation (1), so we should put the sphere at the origin of z when we construct the bijection. However, that argument is very weak because we can use the real part of Z, call it z, to rewrite the Dirichlet series as

$$\mathcal{D}(\mathrm{z}) = \sum_{N=1}^{\infty} \frac{1}{N^{(\mathrm{z}+1)}} \quad , \quad \text{with} \quad \mathrm{z} > 0, \ \mathrm{z} \in \mathbb{R} \quad , \tag{64}$$

in which case the point $\{z, \mathcal{Z}\} = \{0, 0\}$ is no longer favored. Furthermore, there is a seemingly relevant result in general relativity that says every manifold has at least one point where it is possible to construct a coordinate system such that the metric can be written in canonical form and all its first derivatives vanish. Perhaps the point $\mathcal{Z} = 0$ is the one point in $\mathbb{C}$ where, by symmetry, the interesting features are suppressed. If so, this is important because finding one form where the interesting features vanish does not imply their non-existence; the result says that it is a general property of manifolds that one such point always exists, so if we did not have one point that suppresses our result then that might indicate a problem with our analysis.

Even so, we are still faced with the truth that there is a certain case where our argument fails, and the theory of gravitation may or may not be relevant. There might be some subtle point we have neglected which would imply that this certain case is preferred and the other points are not valid for the purposes we have put them to. That would negate our argument which is based on sending $\{Z, \mathcal{Z}\} = \{-1/2, 1/2\}$ to $\mathcal{O}$. On the other hand, if the point where our result fails simply exists, but is not preferred, then we can consider a similar property of the complex exponential. Let's say we want to prove that $e^{i\theta}$

takes on imaginary values for $\theta \in \mathbb{R}$. We have to choose some value for θ so take $\theta = 2\pi\xi$ which gives

$$e^{2\pi i\xi} = \left(e^{2\pi i}\right)^{\xi} = 1^{\xi} \quad \Longrightarrow \quad \mathrm{Im}\left(e^{i\theta}\right) = 0 \;\; \forall \;\; \theta \;. \tag{65}$$

We are able to choose a specific θ that gives the wrong answer and it seems like choosing the point $\mathcal{Z}=0$ is the same type of outlying exception to the rule. The only reason (that we have noticed) to choose $\mathcal{Z}=0$ as the origin of coordinates on the sphere is that it is also the origin of coordinates in equation (1), but that is only one of an infinite number of ways to write the Dirichlet series. It is true that equation (1) is the most irreducible form of the the Dirichlet series, but why should we favor the irreducible form of $\mathcal{D}$ over the irreducible form of ζ, as in equation (58)?

What does this all mean for the Riemann hypothesis? Consider the first and fourth quadrants of the $\hat{\Phi}^1$ region in figure 11. There is no deviation from radiality near $+\mathrm{Re}\hat{\Phi}^0$ so there must be a gradient of increasing ϕ dependence as one moves toward $\pm\mathrm{Im}\hat{\Phi}^0$. Therefore, the right side of the critical strip will be pulled toward the right more strongly than the critical line, meaning that the line $\mathrm{Re}(\mathcal{Z}) = 1/2$ will not be in the center of the critical strip when it achieves its asymptotic behavior at the $\hat{\Phi}^2$ circle. We have taken as a lemma that the proof of an infinite number of zeros on the critical line is reducible to a symmetry argument that says there are an infinite number of zeros centered in the critical strip, but we see that, beyond the $\hat{\Phi}^1$ circle, the center of the critical strip no longer has $\mathrm{Re}(\mathcal{Z}) = 1/2$ due to the gradient in ϕ dependence. Therefore, if the symmetry reduction of Hardy's argument holds, and the rest of the argument holds, then there will be **zeros whose real parts are not equal to one half**. It is true that these zeros will have to be specified with $\hat{\Phi}^n$, but that is surely allowed if other zeros are specified with $\pm i$.

The only remaining issue is to note that it is possible to squeeze an infinite number of zeros into the part of the critical strip within the $\hat{\Phi}^0$ region so that the gradient in ϕ never affects any of them. Consider this: if there are infinitely many zeros on the critical line, and there are infinity of them in the $\hat{\Phi}^0$ region, then Hardy's result does not strictly imply that there will be yet more zeros in the $\hat{\Phi}^n$ planar annuli. If it can be shown that the infinitieth zero lies inside the $\hat{\Phi}^1$ circle then our argument is good, but if it does not then our argument fails completely. It is known that the density of zeros increases as one goes up the critical strip toward infinity and this would be contradicted if there were no zeros inside the $\hat{\Phi}^1$ region. If the density of zeros is N/L then, obviously, we can select small segments L that contain no zeros to show a semantic contradiction when the density decreases from L_j that contains a zero to L_{j+1} that doesn't. However, if we always consider $|L_{j+1}| \ggg |L_j|$, with L_{j+1} further up than L_j, then the result about increasing density is always true. Therefore, in the planar representation, if we take L_1 as the length of the upper critical strip in the $\hat{\Phi}^0$ planar region, and L_2 as the length that crosses the $\hat{\Phi}^1$ and $\hat{\Phi}^2$ planar annuli, which are both infinitely large with respect to $\hat{\Phi}^0$, then the density in L_2 will have to be greater than the density in L_1 which was not zero. Therefore, there must be zeros of the Riemann zeta function beyond the $\hat{\Phi}_0$ region and this, per the above, implies the existence zeros with $\mathrm{Re}(\bar{\mathcal{Z}}_{\bar{n}}) \neq 1/2$. This is another part of our argument that could be considered "less solid" but, to refute it, the burden of proof would lie elsewhere with those who want to demonstrate that the density of zeros eventually stops increasing and then decreases to zero so that there are no zeros beyond $\hat{\Phi}^0$. In the absence of such a demonstration, we cite the general form of the result that says the density of non-trivial zeros increases forever as one goes up the critical strip.

Therefore we have made a good argument against the Riemann hypothesis and likely resolved the Banach–Tarski paradox along the way. As a final note, it may prove useful in the future to label the transfinite annuli and infinitesimal discs, which are also annuli, with oppositely signed $\hat{\Phi}^{\pm n}$.

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[1] Jonathan W. Tooker. Infinitely Complex Topology Changes with Quaternions and Torsion. *viXra:1505.0131*, (2015).
[2] Jonathan W. Tooker. Ontological Physics. *viXra:1312.0168*, (2013).
[3] Jonathan W. Tooker. Quantum Gravity. *viXra:1506.0055*, (2015).
[4] Jonathan W. Tooker. Tempus Edax Rerum. *viXra:1209.0010*, (2012).
[5] Jonathan W. Tooker. Quantum Structure. *viXra:1302.0037*, (2013).
[6] Jonathan W. Tooker. Geometric Cosmology. *viXra:1301.0032*, (2013).
[7] Jonathan W. Tooker. Infinitudinal Complexification. *viXra:1608.0234*, (2016).