Zeros of the Riemann Zeta Function within the Critical Strip and off the Critical Line

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Abstract

In a recent paper, the author demonstrated the existence of real numbers in the neighborhood of infinity. It was shown that the Riemann zeta function has non-trivial zeros in the neighborhood of infinity but none of those zeros lie within the critical strip. While the Riemann hypothesis only asks about non-trivial zeros off the critical line, it is also an open question of interest whether or not there are any zeros off the critical line yet still within the critical strip. In this paper, we show that the Riemann zeta function does have non-trivial zeros of this variety. The method used to prove the main theorem is only the ordinary analysis of holomorphic functions. After giving a brief review of numbers in the neighborhood of infinity, we use Robinson's non-standard analysis and Eulerian infinitesimal analysis to examine the behavior of zeta on an infinitesimal neighborhood of the north pole of the Riemann sphere. After developing the most relevant features via infinitesimal analysis, we will proceed to prove the main result via standard analysis on the Cartesian complex plane without reference to infinitesimals.

§1 Background

Definition 1.1 The Riemann ζ function is the analytic continuation of the Dirichlet series to a meromorphic function on the entire complex plane. In the region $\mathrm{Re}(z)>1$, ζ has the simple form

$$\zeta(z) = \sum_{n=1} \frac{1}{n^z} \quad .$$

Here we will treat ζ as a holomorphic function so the domain of ζ is continued onto the entire complex plane excepting the pole at $z=1$. This is accomplished by way of Riemann's functional equation [1–14]

$$\zeta(z) = \frac{(2\pi)^z}{\pi} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \quad .$$

Definition 1.2 In any coordinate system, the pole of ζ lying at $z(x,y)=1$ in Cartesian coordinates shall be called Z_1.

Definition 1.3 If the Riemann ζ function is a map $\zeta\colon D\to R$, then

$$D = \mathbb{C} \setminus Z_1 \quad , \qquad \text{and} \qquad R = \mathbb{C} \quad .$$

Theorem 1.4 *There exist an infinite number of zeros of the Riemann ζ function with real parts equal to one half.*

Proof. This theorem was proven by Hardy in 1914 [2,15].

Theorem 1.5 *If $\{\gamma_n\}$ is an increasing sequence containing the imaginary parts of the non-trivial zeros of the Riemann ζ function in the upper complex half-plane, then*

$$\lim_{n\to\infty}\big|\gamma_{n+1}-\gamma_n\big|=0 \quad .$$

Proof. Proof of this theorem follows from a 1924 theorem of Littlewood [2, 16].

Corollary 1.6 *The sequence $\{\gamma_n\}$ is unbounded.*

Proof. Proof of this corollary follows from the holomorphism of ζ together with Theorem 1.5.

Remark 1.7 Theorem 1.5 and Corollary 1.6 will serve as the basis for an important theorem: Theorem 5.4. We will show that $\{\gamma_n\}$ is an unbroken line in the neighborhood of infinity (presented in Section 2.) Then it will follow from the property of holomorphic functions that if their zeros are not isolated on a domain, then the functions are constant on that domain. If $\zeta(z)=0$ everywhere on a line segment containing some γ_n, then the main result of this paper will be proven because points in a line are not isolated. We will also show that the non-isolated zeros in the neighborhood of infinity are effectively isolated from the region where ζ is non-constant. Therefore, there will be no contradiction arising between the constancy of ζ on a patch containing some γ_n and the non-constancy of ζ near the origin.

Definition 1.8 If a complex number is expressed in Cartesian coordinates, then $z=z(x,y)$. If it is expressed in plane polar coordinates, then $z=z(r,\theta)$. These numbers are denoted as $z\in\mathbb{C}$. We will use the symbol $\mathbb{C}$ only to refer to the planar representation of all complex numbers.

Definition 1.9 Via what is called the Riemann sphere [17], it is possible to express complex numbers in spherical polar coordinates. In these coordinates, we write $z=z(\phi,\theta)$ and denote them $z\in\Sigma$.

Definition 1.10 The 2-sphere $\mathbb{S}^2$ is charted in spherical polar coordinates with azimuth $\theta\in[0,2\pi)$ and zenith $\phi\in[0,\pi]$.

Definition 1.11 The point $\mathcal{N}\in\mathbb{S}^2$ is given by $\phi=\pi$ and shall be called the north pole. The rest of the sphere shall be called Σ. Therefore,

$$\mathbb{S}^2 = \Sigma \cup \mathcal{N} \quad .$$

Definition 1.12 Although there exist many stereographic projections between $\mathbb{C}$ and Σ, here we will use the convention that $r=0$ when $\phi=0$. The projection functions are

$$\begin{aligned} f:\mathbb{C}\to\Sigma \quad &, \qquad \text{s.t.} \qquad f(r,\theta) = \big(2\tan^{-1} r,\theta\big) \\ f^{-1}:\Sigma\to\mathbb{C} \quad &, \qquad \text{s.t.} \qquad f^{-1}(\phi,\theta) = \left(\tan\frac{\phi}{2},\theta\right) \quad . \end{aligned}$$

Remark 1.13 In the convention of Definition 1.12, $\mathbb{S}^2=\Sigma\cup\mathcal{N}$ is a unit sphere bisected at its equator by $\mathbb{C}$, and centered on the origin of $\mathbb{C}$. The north pole $\mathcal{N}$ lies one unit above the origin of $\mathbb{C}$ and the south pole one unit below. (The south pole is given by $\phi=0$.) Note that the zenith angle $\phi=\pi$ is neither in the range of f nor in the domain of f^{-1} because no complex number $z\in\mathbb{C}$ is projected onto the point $\mathcal{N}$.

Axiom 1.14 The stereographic projection of any infinite straight line in $\mathbb{C}$ onto Σ is a punctured circle passing through the point $\mathcal{N}$, and punctured at that point. If we project the line together with its endpoints at infinity, then the projection is an entire circle on $\mathbb{S}^2$ passing through $\mathcal{N}$.

§2 The Neighborhood of Infinity

Axiom 2.1 The real numbers are an ordered set expressed in interval notation as

$$\mathbb{R} = (-\infty,\infty) \quad .$$

Remark 2.2 The existence of real numbers in the neighborhood infinity, and by proxy complex numbers in the neighborhood of infinity through the definition

$$\mathbb{C} = \big\{x+iy \mid x,y\in\mathbb{R}\big\} \quad ,$$

is proven in References [18, 19]. Therein, the main properties of such numbers are given. For convenience, here we will briefly develop such numbers in Example 2.3 and then we will use them moving forward since their existence is proven and their properties are given elsewhere [18, 19].

Example 2.3 Suppose the interval $x'\in[0,\frac{\pi}{2}]$ consists of all points on some Euclidean line segment AB [20]. x' is a chart covering AB. Define a conformal chart

$$x=\tan(x')\qquad \text{s.t.}\qquad x:\left(0,\frac{\pi}{2}\right)\to(0,\infty)\quad .$$

Over the interior points of AB, we have $x\in\mathbb{R}^+$. Namely, every positive real number $x\in\mathbb{R}^+$ is a cut in the line segment AB. Since every cut in AB is a real number in the x' chart, we also have the corollary property that every cut in AB is a real number in the x chart: $x\in\mathbb{R}^+$. (Throughout this paper, the superscript $+$ indicates the positive-definite subset.)

Assume that every real number less than some natural number is formally constructed by Cauchy sequences in the usual way [21]. (For the present development, it is required to eschew the Dedekind construction of $\mathbb{R}$ in favor of the Cauchy construction.) Suppose b is some real number near the point A where we have $x(A)=0$. Further suppose that $b<n$ for some $n\in\mathbb{N}$. By the mirror symmetry obvious in the geometry of line segments, AB is invariant under permutations of the labels of its endpoints. Since we may permute the endpoints without invoking any contradictions, define an operator $\hat{N}_{CP}$ such that

$$\hat{N}_{CP}(AB)=BA\quad .$$

Under the action of $\hat{N}_{CP}$, the number b near $x(A)=0$ is now another number b' near $x(B)=\infty$. Define

$$\hat{N}_{CP}(x+y)=\hat{N}_{CP}(x)+\hat{N}_{CP}(y)\qquad \text{s.t.}\qquad \begin{cases}\hat{N}_{CP}(0)=\widehat{\infty}\\ \hat{N}_{CP}(x)=-x\quad \text{for}\quad 0<x<\infty\\ \hat{N}_{CP}(\widehat{\infty})=\infty\end{cases}\quad .$$

To determine b', write $b=0+b$. Then operate with $\hat{N}_{CP}$ to obtain

$$b'=\hat{N}_{CP}(0+b)=\hat{N}_{CP}(0)+\hat{N}_{CP}(b)=\widehat{\infty}-b\quad .$$

Under the symmetric permutation of the labels A and B, the number b lying b units away from $x=0$ becomes a number b' in the neighborhood of infinity lying b units away from ∞. With the formal ordering of $\mathbb{R}$ given in References [18, 19], and by Axiom 2.1 giving $\mathbb{R}^+=(0,\infty)$, it is obvious that b' is a real number because

$$(0,\infty)=(0,b']\cup(b',\infty)=(0,\widehat{\infty}-b]\cup(\widehat{\infty}-b,\infty)\quad .$$

Therefore, $b'=\widehat{\infty}-b$ is an ordinary real number with a perfectly rigorous construction given by $\hat{N}_{CP}$ acting on a Cauchy equivalence class.

The symbol $\widehat{\infty}$ has the property $\|\widehat{\infty}\|=\infty$ where

$$\pm\infty=\lim_{x\to0^\pm}\frac{1}{x}\quad ,$$

in the usual way. The hat on infinity is an instruction that tells us not to do the additive or multiplicative absorptive operations on $x\in\mathbb{R}^+$ while the hat is in place. The absorptive operations are

$$\infty \pm x = \infty \quad , \qquad \text{and} \qquad \pm x\infty = \pm\infty \quad .$$

The convenient $\widehat{\infty}$ notation allows us to express numbers like b', called numbers in the neighborhood of infinity, in terms of Cauchy sequences.

The symbol $\widehat{\infty}$ has what is called the non-contradiction property [18, 19]. This property gives the operator $\hat{N}_{CP}$ its name. For $\widehat{\infty}$ to have this property means that it is vested with an innate instance of $\hat{N}_{CP}$. If any contradiction is obtained from the non-absorptivity of $\widehat{\infty}$, then $\widehat{\infty} \to \hat{N}_{CP}(\widehat{\infty}) \to \infty$ and the absorptive operations are restored. This guarantees the robustness and perfect rigor of the analytical framework in the following way. For any real-valued function $f : \mathbb{R} \to \mathbb{R}$ depending on $\hat{N}_{CP}$ (meaning any analytical expression containing $\widehat{\infty}$), if f is used to derive some contradiction, then the non-contradiction property of $\widehat{\infty}$ will be such that

$$f : \mathbb{R} \to \mathbb{R} \quad , \qquad \text{becomes} \qquad f : \mathbb{R} \to \infty \quad .$$

In all such cases, since $f : \mathbb{R} \not\to \mathbb{R}$, it is not possible to use $\widehat{\infty}$ to obtain contradictions within the realm of real analysis. The non-contradiction operator enforces a total ban on all possible contradictions that might arise as a result of choosing not to do the absorptive operations [18, 19]. While simply giving a Cauchy sequence definition for b' is a nice result on its own, the main purpose of the $\widehat{\infty}$ notation [18, 19] is to facilitate arithmetic operations of the form

$$\big(\widehat{\infty} - b\big) - \big(\widehat{\infty} - a\big) = a - b \quad .$$

Without the hat, the expression on the left is undefined by $\infty - \infty$ though the expression on the right clearly follows from a, b (or a', b') being two real numbers near an endpoint of AB.

To restate for clarity before moving on, we will show the principle without the $\widehat{\infty}$ notation and thereby demonstrate the underlying principle. Suppose the operator which permutes the labels A and B is $\hat{N}^2_{CP}$. Then

$$\hat{N}^2_{CP}(0 - b) = \hat{N}^2_{CP}(0) + \hat{N}^2_{CP}(-b) = \hat{N}_{CP}(\infty) - b \quad .$$

Now suppose $\hat{N}_{CP}(\infty) = \infty$ and that the $\hat{N}_{CP}$ operator is an instruction not to do any absorptive operations with ∞ while $\hat{N}_{CP}$ is in place. Since there is some freedom either to do these absorptive operations or not inherent in the order of algebraic operations, the non-contradiction operator does not invoke any contradictions on its own. However, if $\hat{N}^2_{CP}$ is used to construct numbers in the neighborhood of infinity, and those numbers are then used to derive a contradiction, then we are forced to operate as $\hat{N}_{CP}(\infty) \to \infty$. Then the absorption kicks in and

$$(0,\infty) \neq (0,\infty] \cup (\infty,\infty) \quad ,$$

kicks the whole thing out of the realm of real analysis. With all of this in mind, it is better to move the redundant second instance of $\hat{N}_{CP}$ into the hat on $\widehat{\infty}$, and then begin the analysis with a single instance of $\hat{N}_{CP}$. Overall, the non-contradiction operator which constructs real numbers in the neighborhood of infinity cannot be used to derive any contradictions. $\hat{N}_{CP}$ is non-contradictory!

Definition 2.4 We have suppressed the multiplicative absorption of $\widehat{\infty}$ so we will introduce the notation for $0 \leq x \leq 1$

$$x \cdot \widehat{\infty} = \aleph_x \quad .$$

Remark 2.5 Although such numbers as $\aleph_x$ might look foreign to some, we will show in Section 4 that no less a mathematician than Euler used the number $\frac{\infty}{2} \sim \aleph_{0.5}$ in his most seminal works. Euler's analysis of infinities and infinitesimals has been deemed insufficient by the modern establishment [22] but such numbers as $\aleph_x$ cannot lead to contradictions in standard analysis because $\widehat{\infty}$ is vested by the non-contradiction property with an innate instance of the non-contradiction operator. If a contradiction is obtained with the $\aleph_x$ notation, then it is ejected from the realm of standard analysis by

$$\aleph_x \to x\widehat{\infty} \to x\hat{N}_{CP}(\widehat{\infty}) \to x\infty \to \infty \quad .$$

Definition 2.6 Following Axiom 2.1, the set of all real numbers is

$$\mathbb{R} = \Big\{ x \mid -\infty < x < \infty \Big\} \quad .$$

Definition 2.7 The set of real numbers in the neighborhood of the origin is

$$\mathbb{R}_0 = \Big\{ \, x \mid (\exists n \in \mathbb{N})[-n < x < n] \Big\} \quad .$$

Here we define $\mathbb{R}_0$ as the set of all x such that there exists an $n \in \mathbb{N}$ allowing us to write $-n < x < n$.

Definition 2.8 The set of all real numbers in the neighborhood of infinity is

$$\mathbb{R}_\infty = \mathbb{R} \setminus \mathbb{R}_0 \quad .$$

Definition 2.9 The set of large real numbers in the neighborhood of infinity is

$$\widehat{\mathbb{R}} = \Big\{ \pm\big(\widehat{\infty} - b\big) \mid b \in \mathbb{R}_0^+ \Big\} \quad .$$

Remark 2.10 Although numbers having magnitudes greater than any $x\in\mathbb{R}_0$ and less than any $x\in\widehat{\mathbb{R}}$ are not used in the present analysis, namely numbers of the form $x=\aleph_\mathcal{X}\pm b$ with $0<\mathcal{X}<1$ and $b\in\mathbb{R}_0$, it must be noted that

$$\mathbb{R}_\infty\setminus\widehat{\mathbb{R}}\neq\emptyset \quad .$$

For this reason, $\widehat{\mathbb{R}}$ is called the set of *large* real numbers in the neighborhood of infinity. Numbers in the set $\mathbb{R}_\infty\setminus\widehat{\mathbb{R}}$ are treated most specifically in Reference [19]. To be perfectly explicit, if x,y are positive numbers such that $x\in\mathbb{R}_0$ and $y\in\widehat{\mathbb{R}}$, then $x<y$.

Definition 2.11 The complex neighborhood of the origin is

$$\mathbb{C}_0=\Big\{re^{i\theta}\mid r\in\mathbb{R}_0,\ \theta\in\mathbb{R}_0\Big\} \quad .$$

Definition 2.12 The provisional large complex neighborhood of infinity is

$$\widehat{\mathbb{C}}=\Big\{re^{i\theta}\mid r\in\widehat{\mathbb{R}},\ \theta\in\mathbb{R}_0\Big\} \quad .$$

Remark 2.13 Definition 2.12 is called the "provisional" large complex neighborhood of infinity because it assumes a single complex infinity. As the neighborhood of infinity has been developed in the present framework of analysis [18, 19], the proper definition would be

$$\widehat{\mathbb{C}}=\Big\{x+iy\mid x,y\in\mathbb{R},\ (x\in\widehat{\mathbb{R}})\vee(y\in\widehat{\mathbb{R}})\vee(x,y\in\widehat{\mathbb{R}})\Big\} \quad ,$$

where $\vee$ means "or." Due to the arithmetic of $\widehat{\mathbb{R}}$ [18,19], the usual conversion between Cartesian and polar coordinates gives

$$\begin{aligned}
z(x,y)&=\big(\widehat{\infty}-b\big)\pm iy &&\implies && \theta=0 \quad \forall b,y\in\mathbb{R}_0^+\\
z(x,y)&=\big(\widehat{\infty}-b\big)+i\big(\widehat{\infty}-a\big) &&\implies && \theta=\frac{\pi}{4} \quad \forall b,a\in\mathbb{R}_0^+\\
z(x,y)&=\pm x+i\big(\widehat{\infty}-a\big) &&\implies && \theta=\frac{\pi}{2} \quad \forall x,a\in\mathbb{R}_0^+ \quad ,
\end{aligned}$$

and

$$z(x,y)=\big(\aleph_\mathcal{X}-b\big)+i\big(\aleph_\mathcal{Y}-a\big) \quad\implies\quad \theta=\tan^{-1}\left(\frac{\mathcal{Y}}{\mathcal{X}}\right) \quad \forall b,a\in\mathbb{R}_0^+ \quad .$$

The one-to-one correspondence between $z(x,y)$ and $z(r,\theta)$ is lost in the neighborhood of infinity. However, for the purposes of the analysis of the holomorphic function ζ, Definition 2.12 will be presently sufficient. We will make a remark on the relevance of the distinction at the end of Section 4.

Theorem 2.14 *For any real-valued zenith angle $\phi \in [0,\pi)$ of $\mathbb{S}^2$, the inverse projection of $z \in \Sigma$ onto $\mathbb{C}$ by $f^{-1}\!:\!\Sigma \to \mathbb{C}$ is a number of the form $z(r,\theta)$ such that $r \in \mathbb{R}_0$. In other words, when $\phi \in \mathbb{R}$, we have*

$$f^{-1} : \Sigma \to \mathbb{C}_0 \quad .$$

Proof. Restrict the domain of the tangent function to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose $\{\beta_n\}$ is a monotonic increasing sequence of real numbers with the property that $\tan(\beta_n) = n$ for every $n \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} \beta_n = \frac{\pi}{2} \quad .$$

Under the given condition that $\phi \in \mathbb{R}$, we have

$$\forall \phi \in [0,\pi) \quad \exists \beta_n \in \{\beta_n\} \qquad \text{s.t.} \qquad \beta_n > \frac{\phi}{2} \quad .$$

We obtain from the monotonic behavior of the tangent on the domain $(-\frac{\pi}{2}, \frac{\pi}{2})$

$$\forall \phi \in [0,\pi) \quad \exists \beta_n \in \{\beta_n\} \qquad \text{s.t.} \qquad \tan(\beta_n) > \tan\left(\frac{\phi}{2}\right) \quad .$$

Since $\tan(\beta_n)$ is some natural number $n \in \mathbb{N}$ and $r = \tan(\frac{\phi}{2})$ (Definition 1.12), we find that

$$n > r \qquad \Longleftrightarrow \qquad r \in \mathbb{R}_0 \quad .$$

Axiom 2.15 The range of ζ does not exceed the neighborhood of the origin, namely

$$\zeta : \mathbb{C} \setminus Z_1 \to \mathbb{C}_0 \quad .$$

§3 Non-standard Analysis

Remark 3.1 Although we will not prove with hyperreal numbers the existence of the zeros of ζ which are the main result of this paper, in this section we will use Robinson's non-standard analysis [23–29] to put the relevant qualitative features on a rigorous foundation and to flesh out some of the fine nuance. In Section 4, we will make a classically standard but heuristic argument in favor of the existence of the zeros, and then in Section 5 we will rigorously prove the existence of the zeros with standard analysis.

Definition 3.2 The hyperreal number system $*\mathbb{R}$ [23–29] contains an infinite element ω, called in the jargon an unlimited element, such that $\omega > x$ for any $x \in \mathbb{R}$.

Definition 3.3 The hyperreal number system $*\mathbb{R}$ [23–29] contains an infinitesimal element ε such that

$$\varepsilon = \frac{1}{\omega} \ , \qquad \text{and} \qquad 0 < \varepsilon < x \quad \forall x \in \mathbb{R}^+ \ .$$

Remark 3.4 The arithmetic operations of hyperreal numbers may be found in References [23–29].

Definition 3.5 Two hyperreal numbers $x, y \in *\mathbb{R}$ are said to be close if $x = y$ or if $|y - x|$ is an infinitesimal quantity. Closeness is denoted as

$$\forall z \in \mathbb{R}^+ \quad \exists |y - x| < z \qquad \Longleftrightarrow \qquad x \simeq y \ .$$

The quantity $|y - x|$ is called the distance between x and y.

Definition 3.6 The standard part of $x \in *\mathbb{R}$ is the unique $x_0 \in \mathbb{R}$ such that $x \simeq x_0$.

Definition 3.7 The halo of a point P is the set of all points which are close to P. The halo is denoted

$$hal(P) = \{ x \in *\mathbb{R} \mid x \simeq P \} \ .$$

Definition 3.8 If P is a point in M, then $hal(P) \cap M$ is called the halo of P in M. This is denoted

$$hal_M(P) = M \cap hal(P) \ .$$

Definition 3.9 Although Riemann himself likely used a definition of the unit 2-sphere $\mathbb{S}^2$ in the form

$$\mathbb{S}^2_{\text{Riemann}} = \mathbb{S}^2_{\mathbb{R}} = \{ \vec{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \ \} \ ,$$

here we will use without loss of generality

$$\mathbb{S}^2 = \{ \vec{x} \in *\mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \ \} \ .$$

The spherical polar coordinates of $\mathbb{S}^2$ are explicitly $\phi, \theta \in *\mathbb{R}$ with bounds $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$. If we refer to the polar angles of $\mathbb{S}^2_{\mathbb{R}}$, then they are strictly real-valued but have the same bounds as the angles on $\mathbb{S}^2$.

Remark 3.10 In order to consider the stereographic projection of the provisional large complex neighborhood of infinity $\widehat{\mathbb{C}}$ onto $\mathbb{S}^2$, we will need to use infinitesimals which do not exist in $\mathbb{R}^3$. This follows from Theorem 2.14 which proved that for any $\Delta\phi \in \mathbb{R}$, the points of Σ having zenith angle $\phi = \pi - \Delta\phi$ will be projected by f^{-1} into $\mathbb{C}_0$. We will show in Theorem 3.11 that it is not possible to use the projection function f to map $\mathbb{C}$ onto $\mathbb{S}^2_{\mathbb{R}}$. If we want to map the entire complex plane onto a unit 2-sphere, then the non-standard $\mathbb{S}^2$ of Definition 3.9 will allow us to do so.

Theorem 3.11 *The stereographic projection function f (Definition 1.12) sends all of $\widehat{\mathbb{C}}$ into $hal_\Sigma(\mathcal{N})$.*

Proof. Let $\Delta\phi \in \mathbb{R}$ be $0 < \Delta\phi \ll \pi$ and let $\phi_0 = \pi - \Delta\phi$. We have proven in Theorem 2.14 that f^{-1} sends every $z(\phi,\theta) = (\phi_0, \theta_0)$ to $z(r,\theta) = (R_0, \theta_0)$ for some $R_0 \in \mathbb{R}_0$. It is known that when the domain of f is extended to $r = \widehat{\infty}$, complex infinity $\widehat{\infty}$ will be mapped onto the point $\mathcal{N}$. The ordering of $\mathbb{R}$ [18, 19] is such that $x < y < \infty$ for any $x \in \mathbb{R}_0$ and $y \in \widehat{\mathbb{R}}$ so by the monotonic behavior of f, all $z \in \widehat{\mathbb{C}}$ must be mapped to a point $z \in \Sigma$ whose zenith angle is less than π yet greater than any $\phi = \pi - \Delta\phi$. By Definition 3.3, such zenith angles are of the form $\phi = \pi - \delta\phi$ where $\delta\phi \in {*\mathbb{R}}$ is an infinitesimal angle $\delta\phi := \varepsilon$. The distance between two points of a unit sphere is

$$s = \psi \quad ,$$

with ψ being the angle between the two points. The angle between $\mathcal{N}$ and any point in Σ having zenith angle $\phi = \pi - \delta\phi$ is

$$\pi - (\pi - \delta\phi) = \delta\phi \quad .$$

It follows that $s = \delta\phi$ is an infinitesimal distance. It follows from Definition 3.8 that f sends all of $\widehat{\mathbb{C}}$ into $hal_\Sigma(\mathcal{N})$.

Corollary 3.12 *For any $b \in \mathbb{R}_0$, the stereographic projection of a line given by $z = \pm\big(\widehat{\infty} - b\big)$ or $z = \pm i\big(\widehat{\infty} - b\big)$ lies entirely within a circle of infinitesimal radius on $\mathbb{S}^2$. Each circle contains $\mathcal{N}$ and each line covers an entire circle except for the point $\mathcal{N}$.*

Proof. It follows from Axiom 1.14 that under the map f each line becomes a circle punctured at $\mathcal{N}$. Since each line lies entirely within $\widehat{\mathbb{C}}$, it follows from Theorem 3.11 that the circle lies entirely within $hal_\Sigma(\mathcal{N})$. All distances between points in that halo are infinitesimal so the radius of the circle is infinitesimal.

Remark 3.13 Figure 1 shows a top view of $\mathbb{S}^2$ looking down on $\mathcal{N}$ toward the south pole. Pictured is an infinitesimal spherical cap containing the stereographic projection of the lines considered in Corollary 3.12, as well as the

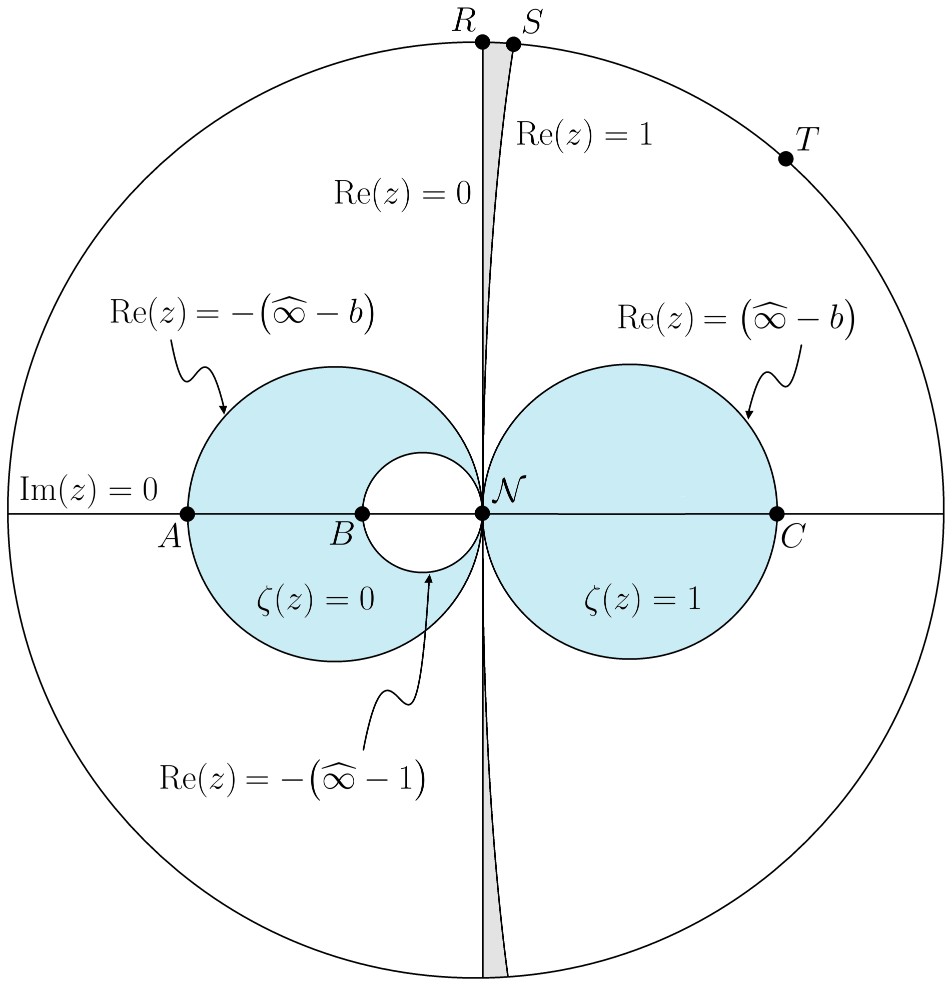

Figure 1: This figure shows a portion of $hal_{\Sigma}(\mathcal{N})$. The real axis of $\mathbb{C}$ is the line passing through A and C, and the imaginary axis through R and $\mathcal{N}$. These axes appear as straight lines in this figure because they are great circles of $\mathbb{S}^2$. The circle passing through S and $\mathcal{N}$ (which does not lie entirely within $hal_{\Sigma}(\mathcal{N})$) is the planar line $\text{Re}(z)=1$. The critical strip is shaded in gray (not to scale.) The circle passing through A and $\mathcal{N}$ is the planar line $\text{Re}(z)=-(\widehat{\infty}-b)$ for some $b\in\mathbb{R}_0>1$. The circle passing through B and $\mathcal{N}$ is the planar line $\text{Re}(z)=-(\infty-1)$ and we have shown in Reference [18] that $\zeta=0$ everywhere in the region between the circles $A\mathcal{N}$ and $B\mathcal{N}$. The circle passing through $\mathcal{N}$ and C is the planar line $\text{Re}(z)=(\widehat{\infty}-b)$. We have shown in Reference [18] that $\zeta=1$ everywhere inside this circle. Note very well: although the circle RST is infinitesimal in radius, everything other than the point $\mathcal{N}$ belongs to Σ because we have stated in Definition 1.11 that $\Sigma=\mathbb{S}^2\backslash\mathcal{N}$.

critical strip (thin gray region) and the real axis of $\mathbb{C}$. We have proven in Reference [18] that $\zeta(z)$ is equal to zero everywhere in the leftward blue region and that it is equal to one everywhere in the rightward blue region. According to the definition of the Riemann hypothesis given by the Clay Mathematics Institute [30], we have shown that the hypothesis is false on account of the zeros in the leftward blue region.

Theorem 3.14 *The characteristic scale of the features depicted in Figure 1 is of at most order*

$$\mathcal{O}(s) = \mathcal{O}\left(\lim_{n\to\omega} \varepsilon^n\right) \quad ,$$

where s is the arc length.

Proof. Consider a polar ray of $\mathbb{S}^2$ anchored at $\mathcal{N}$ which sweeps out the positive real axis of $\mathbb{C}$ starting at $z=0$. We have proven in Theorem 2.14 that any ray passing though a real-valued zenith angle $\phi\in[0,\pi]$ will pass through the real axis in $\mathbb{R}_0$. Using the notation of Definition 2.4, before the ray can reach $\widehat{\mathbb{R}}$ it must sweep through an infinite number of neighborhoods of the form

$$\mathbb{R}_{\mathcal{X}} = \Big\{ \pm\big(\aleph_{\mathcal{X}} \pm b\big) \mid b \in \mathbb{R}_0,\ 0 < \mathcal{X} < 1 \Big\} \quad .$$

It follows from Axiom 2.1 that $\mathbb{R}$ is connected so as the ray leaves $\mathbb{R}_0$ it will enter the least neighborhood of real numbers greater than $\mathbb{R}_0$. An extension of Theorem 2.14 would show that for any $\phi_0=\pi-a\varepsilon$, with $a\in\mathbb{R}_0^+$ some absolute constant, the polar ray intersects the real axis in a neighborhood of $\mathbb{R}$ adjacent to $\mathbb{R}_0$. By a recursive argument, this theorem is proven.

Remark 3.15 Here we have come to the end of what we may easily demonstrate with Robinson's $*\mathbb{R}$. Since $\varepsilon^2\in *\mathbb{R}$ is simply an infinitesimal, and it is not in any rigorous sense infinitesimal with respect to ε, there is no guarantee that we could actually project all of $\mathbb{C}$ onto Σ. In strict hyperreal analysis, we might be limited to $\mathbb{C}_0$ together with the least $\mathbb{C}_{\mathcal{X}}$ neighborhood as the largest portion of $\mathbb{C}$ which f can send onto $\mathbb{S}^2$ as given by Definition 3.9. We may use f to rigorously construct $hal_\Sigma(\mathcal{N})$ as in Figure 1 if we set the domain of f as $\mathbb{C}_0\cup\widehat{\mathbb{C}}$. This defines $\widehat{\mathbb{C}}$ as an adjacent (though disconnected) neighborhood to $\mathbb{C}_0$. However, the proof of Main Theorem 5.5 is simple enough that we are not motivated to go down the path required for such a workaround.

As a last aside before moving on, we show in Reference [19] that the connected property of $\mathbb{R}$ requires a Cantor set of points or neighborhoods between sequentially greater $\mathbb{R}_{\mathcal{X}}$ neighborhoods. As the polar ray leaves $\mathbb{R}_0$, it must intersect $\mathbb{R}$ at one or more of these Cantor numbers before the ray could enter the least $\mathbb{R}_{\mathcal{X}}\in\mathbb{R}_\infty$. We will not needlessly complicate the main result of this paper by treating the Cantor set here.

§4 Eulerian Analysis

Remark 4.1 In this section, we will analyze ζ in the Leibniz–Euler–Cauchy (LEC) tradition of infinitesimal mathematics [22]. Although this approach is not sufficient for demonstrations at the level of Bolzano–Weierstrass in real analysis or the properties of holomorphic functions in complex analysis, here we will set the heuristic stage for exactly that approach to appear in Section 5.

Definition 4.2 For the purposes of Eulerian analysis alone, the letter i refers to an infinitely large integer such that $i > n$ for any $n \in \mathbb{N}$.

Definition 4.3 The Eulerian infinitesimal is ϵ. It has the property

$$\epsilon = \frac{1}{i} \quad , \qquad \text{and} \qquad 0 < \epsilon < \frac{1}{n} \quad \forall n \in \mathbb{N} \quad .$$

Remark 4.4 Although Robinson's infinitesimals are not such that there exist quantities which are "infinitesimal even compared to other infinitesimals," there do exist such Eulerian infinitesimals. In References [31, 32] (treated in Reference [22]) Euler writes the following about some $x \in \mathbb{R}$.

> "[*The quantity*] x^2/i^2 can be ignored because even when multiplied by i it remains infinitely small."

Therefore, for any two Eulerian infinitesimals $\epsilon^n \neq \epsilon^m$, ϵ^n is said to be infinitesimal with respect to ϵ^m whenever $n > m$. In Section 3, it was not perfectly obvious we would be able to project all of $\mathbb{C}$ onto Σ due the absence of tiered infinitesimality in Robinson's framework of analysis. With the LEC infinitesimal, however, we do have the requisite tiered structure.

Example 4.5 The utility of the LEC infinitesimal is seen via the antiquated definition of the derivative

$$\frac{d}{dx} f(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon} \quad .$$

For example, Euler would have calculated the derivative of $f(x) = 3x^2$ as

$$\frac{d}{dx} f(x) = \frac{3\big(x+\epsilon\big)^2 - 3x^2}{\epsilon} = \frac{3\big(x^2 + 2x\epsilon + \epsilon^2\big) - 3x^2}{\epsilon} = 6x + \epsilon \quad .$$

By virtue of ϵ being infinitesimal, Euler would have found $\frac{d}{dx} f(x) = 6x$ due to the following principle originally written down by Euler himself [22, 33, 34].

> "[*There exists a*] well-known rule that the infinitely small vanishes in comparison with the finite and hence can be neglected with respect to it."

To satisfy the requirement that the derivative of a real-valued function is also real-valued, Euler implicitly takes what Robinson has called the "standard part" of $\frac{d}{dx}f(x)$ (Definition 3.6.) Of course, Euler's method gives the exact same derivative as the modern formula

$$\frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad .$$

Remark 4.6 Almost everything Euler proved with infinitesimals has been reproven to modern standards of rigor in the hyperreals or through the ε–δ formalism of Bolzano and Weierstrass (and Cauchy) [22]. The LEC approach is usually known to produce correct results, albeit via a path of argument that is insufficient according to certain modern standards of rigor. It is because the results of such analyses can usually be trusted that we include the present section dedicated to Eulerian analysis before giving the formal proof in Section 5.

As in the previous section, it is our present goal to study the behavior of ζ near $\mathcal{N}$. The radius of $\mathbb{S}^2$ contributes to the form of the stereographic projection functions (Definition 1.12) but the radius does not contribute to the spherical polar coordinates $z \in \Sigma$ that we have established with $f : \mathbb{C} \to \Sigma$. Since the one-to-one correspondence between $z \in \mathbb{C}$ and $z \in \Sigma$ may be preserved for any radius R of $\mathbb{S}^2$, we will treat R a scale factor allowing us to zoom in on $\mathcal{N}$ without introducing the notion of a halo.

Definition 4.7 As the radius of the sphere has no relevance beyond the specific form of the stereographic projections, redefine $\mathbb{S}^2$ as

$$\mathbb{S}^2 = \{ \vec{x} \in \mathbb{R}^3_{\text{Euler}} \mid x_1^2 + x_2^2 + x_3^2 = R \in \mathbb{R}^+ \ \} \quad .$$

Definition 4.8 With i being Euler's infinite integer, define Σ_I to be such that

$$\Sigma_I = \lim_{R \to i} \Sigma \quad .$$

Definition 4.9 The Gaussian curvature $K \in \mathbb{R}$ of $\mathbb{S}^2$ is

$$K = \frac{1}{R^2} \ .$$

Theorem 4.10 *In the limit $R \to i$, the Gaussian curvature vanishes on a ϵ-neighborhood of $\mathcal{N}$.*

Proof. We have

$$\lim_{R\to i} K = \lim_{R\to i} \frac{1}{R^2} = \frac{1}{i^2} = \epsilon^2 \quad .$$

Following Example 4.5, the Gaussian curvature vanishes.

Definition 4.11 Theorem 4.10 motivates us to establish a plane polar coordinate chart $(r_\mathcal{N}, \theta_\mathcal{N})$ on an ϵ-neighborhood of $\mathcal{N}$. The point $\mathcal{N}$ is given by $r_\mathcal{N} = 0$ and the azimuth is usual one $\theta_\mathcal{N} = \theta$. These coordinates are denoted $z \in \Sigma_I$ although $r_\mathcal{N} = 0$ is not properly in Σ_I.

Axiom 4.12 For any two points z_1, z_2 in the domain of the Riemann ζ function, $\zeta(z_1) = \zeta(z_2)$ whenever $|z_2 - z_1| = 0$.

Example 4.13 In this example, we will use the LEC infinitesimal to prove, heuristically, that ζ has zeros off the critical line yet within the critical strip. The distance between two points represented in plane polar coordinates is

$$d(z_1, z_2) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)} \quad .$$

We will use this formula to study distance in the ϵ-neighborhood of $\mathcal{N}$ where we have established the $z(r_\mathcal{N}, \theta_\mathcal{N})$ plane polar coordinates. Using the sequence $\{\gamma_n\}$ introduced in Theorem 1.5, we will take as our first point

$$z_1 \in \widehat{\mathbb{C}} \qquad \text{s.t.} \qquad \text{Re}(z_1) = \frac{1}{2} \quad , \quad \text{Im}(z_1) \in \{\gamma_n\} \quad , \qquad \text{and} \qquad \zeta(z_1) = 0 \quad .$$

The point $z_1 \in \widehat{\mathbb{C}}$ is guaranteed to exist by Corollary 1.6. Since we are only making a heuristic argument in this example, we will assume in the limit $R \to i$ that $r_\mathcal{N}$ which was infinitesimal in $hal_\Sigma(\mathcal{N})$ measures finite distance among the major features of Figure 1.

For $z_2(r_\mathcal{N}, \theta_\mathcal{N}) \in \widehat{\mathbb{C}}$, we will pick a point inside the critical strip at the same radius as z_1 but with a slightly lesser $\theta_\mathcal{N}$. The angular separation of z_1, z_2 in the $z(r_\mathcal{N}, \theta_\mathcal{N})$ coordinates is obviously some infinitesimal angle. To prove it, note that we have not drawn the width of the critical strip to scale in Figure 1. The lines $\text{Re}(z) = 1$ and $\text{Re}(z) = (\widehat{\infty} - b)$ are both punctured circles tangent to the imaginary axis at the point $\mathcal{N}$ but in the limit $R \to i$, the radius of $\text{Re}(z) = 1$ goes to infinity meaning that it becomes a straight line. If Figure 1 was drawn to scale, then the lines defining the critical strip would be collinear in the infinitesimal neighborhood of $\mathcal{N}$. The only way to preserve the single point of tangency is to have an infinitesimal angle between the lines $\text{Re}(z) = 0$ and $\text{Re}(z) = 1$. Since z_1 and z_2 are two points inside the critical strip at the same radius $r_\mathcal{N}$, the angular separation between them is some infinitesimal angle. Inserting

$$z_1(r_\mathcal{N}, \theta_\mathcal{N}) = (r_1, \theta_1) \quad , \qquad \text{and} \qquad z_2(r_\mathcal{N}, \theta_\mathcal{N}) = (r_1, \theta_1 - a\epsilon)$$

into the distance formula yields

$$d(z_1,z_2)=\sqrt{2r_1^2-2r_1^2\cos(a\epsilon)}=\sqrt{2}r_1\left(1-1+\frac{(a\epsilon)^2}{2!}-...\right)^{\frac{1}{2}}=\sqrt{2}r_1\mathcal{O}(\epsilon)\quad .$$

Having completed the calculation we should set infinitesimal terms to zero giving $d(z_1,z_2)=0$. By Axiom 4.12, $\zeta(z_2)=0$ so we have demonstrated that there exist zeros off the critical line yet within the critical strip. These zeros are in the neighborhood of infinity.

Remark 4.14 We have chosen in Example 4.13 z_2 within the critical strip. However, since we know that points along the critical line near $\mathcal{N}$ have an azimuth $\theta_{\mathcal{N}}$ only infinitesimally less than $\frac{\pi}{2}$, we could have chosen z_2 on the line $\text{Re}(z)=1$ and the distance computation would have come out the same. This contradicts the theorem of Hadamard and de la Vallée-Poussin [2, 35, 36] which claims that there are no zeros of ζ on that line. When we give the formal proof in Main Theorem 5.5, it will similarly prove the existence of zeros on that line. The discrepancy is certainly that Hadamard and de la Vallée-Poussin did not consider the neighborhood of infinity.

Example 4.15 To demonstrate the robust validity of the Eulerian analytical method, now we will consider z_2 as an internal point of the straight line segment $\mathcal{N}C$ (Figure 1), and then we will compute the distance to z_1 with imaginary part γ_n on the critical line, as in the Example 4.13. We have

$$z_1(r_{\mathcal{N}},\theta_{\mathcal{N}})=\left(r_1,\frac{\pi}{2}-a\epsilon\right)\quad,\qquad\text{and}\qquad z_2(r_{\mathcal{N}},\theta_{\mathcal{N}})=(r_1,0)\quad,$$

with the properties

$$\zeta(z_1)=0\quad,\qquad\text{and}\qquad\zeta(z_2)=1\quad.$$

The distance formula yields

$$d(z_1,z_2)=\sqrt{2r_1^2-2r_1^2\cos\left(\frac{\pi}{2}-a\epsilon\right)}=r_1\sqrt{2+2\sin(a\epsilon)}=\sqrt{2}r_1\big(1+\mathcal{O}(\epsilon)\big)^{\frac{1}{2}}\ .$$

Ignoring terms of order ϵ, we find that $d(z_1,z_2)\neq 0$ and that, therefore, Axiom 4.12 does not apply. We do not contradict the known property $\zeta(z_1)\neq\zeta(z_2)$ and it is demonstrated that the Eulerian analysis is decently robust.

Remark 4.16 Axiom 4.12 does not produce any contradictions anywhere in the Eulerian ϵ-neighborhood of $\mathcal{N}$. However, it is a property of ζ that it is holomorphic on $\mathbb{C}$, and that requires that ζ's zeros are either isolated or that it is constant on the domain. Looking at Figure 1, it is obvious that the zeros of ζ are not isolated in the ϵ-neighborhood of $\mathcal{N}$. The problematic issue is that raised in Remark 2.13: we have the used provisional large complex neighborhood of infinity in the form

$$\widehat{\mathbb{C}} = \left\{ re^{i\theta} \mid r \in \widehat{\mathbb{R}},\ \theta \in \mathbb{R}_0 \right\} \quad .$$

as opposed to the proper definition

$$\widehat{\mathbb{C}} = \{ x + iy \mid x, y \in \mathbb{R},\ (x \in \widehat{\mathbb{R}}) \vee (y \in \widehat{\mathbb{R}}) \vee (x, y \in \widehat{\mathbb{R}}) \} \quad .$$

If we want to use f to project proper $\widehat{\mathbb{C}}$ onto Σ, then the easiest way would be to map $\widehat{\mathbb{C}}$ onto $\mathbb{C}_0$ with some function g, and then use

$$f \circ g : \widehat{\mathbb{C}} \to \mathbb{C}_0 \to \widehat{\Sigma} \quad ,$$

where the hat on $\widehat{\Sigma}$ indicates the pre-image in $\widehat{\mathbb{C}}$. If we restrict $\widehat{\mathbb{C}}$ to allow only the real or imaginary part in $\widehat{\mathbb{R}}$, but not both, and we restrict the non-$\widehat{\mathbb{R}}$ part to $\mathbb{R}_0$, then the function g should contain a translation part g_1 of the form

$$g_1(x, y) = \begin{cases} (\widehat{\infty} \mp x, y) & \text{if} \quad x \in \widehat{\mathbb{R}}^{\pm},\ y \in \mathbb{R}_0 \\ (x, \widehat{\infty} \mp y) & \text{if} \quad x \in \mathbb{R}_0,\ y \in \widehat{\mathbb{R}}^{\pm} \end{cases} \quad .$$

Considering the neighborhoods of $+\widehat{\infty}$ and $+i\widehat{\infty}$ translated by g_1 toward the origin, we see they will overlap in the first quadrant. Therefore, g must also contain a deformation part g_2 to squeeze the neighborhood of $+\widehat{\infty}$ into $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$ and the neighborhood of $+i\widehat{\infty}$ into $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4})$. Through the definition of $\widehat{\mathbb{R}}$ which depends on $(\widehat{\infty} - b)$ with $b > 0$, we see that the translations of the neighborhoods of $+i\widehat{\infty}$ and $+\widehat{\infty}$ will not contain the real and imaginary axes respectively. The deformation map g_2 should avoid the overlap by sending the axes to $\theta = \frac{n\pi}{4}$ for odd n. Therefore, we have a set of topological obstructions in the range of

$$f \circ \big(g_2 \circ g_1 \big) : \widehat{\mathbb{C}} \to \mathbb{C}_0 \to \widehat{\Sigma} \quad .$$

The obstructions preclude any patch in $\widehat{\Sigma}$ containing values of z for which both $\zeta(z) = 0$ and $\zeta(z) = 1$. Therefore, if we take care to project the proper large complex neighborhood of infinity rather than the provisional one, we will not obtain any contradictions with the requirement that ζ is holomorphic on $\widehat{\Sigma}$. When we take the full form of $\widehat{\mathbb{C}}$ which allows both of x and y to range across any $\mathbb{R}_{\mathcal{X}}$, that will only add more topological obstructions. However, as in Section 3, we have gone off on a tangent and made things much more complicated than they need to be.

§5 Standard Analysis

Remark 5.1 We have shown in Theorem 3.11 and Corollary 3.12 that it is not possible to map all of $\mathbb{C}$ onto

$$\mathbb{S}^2_{\mathbb{R}} = \{ \vec{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \ \} \quad .$$

In this section, therefore, we will prove Main Theorem 5.5 directly in the Cartesian coordinates.

Proposition 5.2 *If f is a holomorphic function defined everywhere on an open connected set $D\subset\mathbb{C}$, and if there exists more than one $z_0\in D$ such that $f(z_0)=0$, then f is constant on D or the set containing all $z_0\in D$ is totally disconnected.*

Refutation. This proposition is usually proven by a line of reasoning starting with the following. By the holomorphism of f and the property $f(z_0)=0$, we know there exists a convergent Taylor series representation of $f(z)$ for all $|z-z_0|<R$ with $R\in\mathbb{R}$. Here the proposition fails pseudo-trivially because we can select $R\in\widehat{\mathbb{R}}$ and assume

$$\big(\widehat{\infty}-b\big)<\big|z-z_0\big|<\big(\widehat{\infty}-a\big) \quad ,$$

to show that the Taylor series does not converge. We have

$$f(z)=f(z_0)+f'(z_0)\big(z-z_0\big)+\frac{f''(z_0)}{2!}\big(z-z_0\big)^2+\ldots \quad .$$

The first term in the series vanishes by definition and so, therefore, we have by assumption

$$f(z)>f'(z_0)\big(\widehat{\infty}-b\big)+\sum_{n=2}^{\infty}\frac{f^{(n)}(z_0)}{n!}\big(\widehat{\infty}-b\big)^n \quad .$$

The Taylor series expansion of f does not converge for $|z-z_0|\in\widehat{\mathbb{R}}$. This follows from $(\widehat{\infty}-b)^k=\widehat{\infty}$ for all $k\geq 2$ [18, 19].

Axiom 5.3 If f is a holomorphic function defined everywhere on an open connected set $D\subset\mathbb{C}$, if there exists more than one $z_0\in D$ such that $f(z_0)=0$, and if every $p\in D$ is such that $|z_0-p|\in\mathbb{R}_0$, then f is constant on D or the set containing all $z_0\in D$ is totally disconnected.

Theorem 5.4 *If $\{\gamma_n\}$ is an increasing sequence containing the imaginary parts of the non-trivial zeros of the Riemann ζ function in the upper complex half-plane, then*

$$\lim_{n\to(\widehat{\infty}-b)}\big|\gamma_{n+1}-\gamma_n\big|=0 \quad .$$

Proof. To prove this theorem, we will refer to Theorem 1.5 which follows from a theorem of Littlewood [2, 16]. The exact form of Littlewood's 1924 theorem is

"For every large T, $\zeta(s)$ has a zero $\beta + i\gamma$ satisfying

$$|\gamma - T| < \frac{A}{\log\log\log T} \quad ."$$

The proof of the present theorem follows from Littlewood's result in exactly the same way that Theorem 1.5 follows. For proof by contradiction, assume

$$\lim_{n\to(\widehat{\infty}-b)} \big|\gamma_{n+1} - \gamma_n\big| \neq 0 \quad .$$

Then there exists some $m(n)$ and some $a\in\mathbb{R}_0^+$ such that

$$\lim_{m(n)\to(\widehat{\infty}-b)} \big|\gamma_{m(n)+1} - \gamma_{m(n)}\big| > 2a \quad .$$

Let T_n be the average of $\gamma_{m(n)+1}$ and $\gamma_{m(n)}$ so

$$T_n = \frac{\gamma_{m(n)+1} + \gamma_{m(n)}}{2} \quad .$$

Now we have

$$\lim_{T_n\to(\widehat{\infty}-b)} \big|\gamma - T_n\big| > a \quad ,$$

because T_n is centered between the next greater and next lesser γ_n, and we have shown that they are separated by more than $2a$. This contradicts Littlewood's result

$$|\gamma - T_n| < \frac{A}{\log\log\log T_n} \quad , \qquad \text{whenever} \qquad \frac{A}{\log\log\log T_n} < a \quad .$$

The limit $T_n\to(\widehat{\infty}-b)$ is exactly such a case because the identity $\log(a-b) = \log(a) + \log(1-\frac{b}{a})$ gives

$$\big|\log(\widehat{\infty}-b)\big| = \infty \quad .$$

Therefore, the elements of $\{\gamma_n\}$ form an unbroken line in the neighborhood of $i\widehat{\infty}$ and the present theorem is proven.

Main Theorem 5.5 *The Riemann ζ function has zeros within the critical strip yet off the critical line.*

Proof. Proof follows from Axiom 5.3 and Theorem 5.4.

Remark 5.6 By refuting Proposition 5.2, we have not altered anything fundamental about the properties of holomorphic functions because the proposition holds when $|z-z_0|\in\mathbb{R}_0$. This case encompasses the totality of standard analysis previous to the discovery of numbers in the neighborhood of infinity [18,19].

Presently, $|z-z_0|\in\mathbb{R}_0$ is not a global constraint on any $D\subset\mathbb{C}$ containing both the non-isolated zeros near $\pm i\widehat{\infty}$ and the constant values $\zeta(z)=1$ near $+\widehat{\infty}$. Similarly, all regions on which ζ is constant, meaning the neighborhoods of $\pm\widehat{\infty}$ and $\pm i\widehat{\infty}$, are separated from the non-constant behavior of ζ on $\mathbb{C}_0\setminus Z_1$ by $|z-z_0|\notin\mathbb{R}_0$. Therefore, everything works out perfectly.

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